Einstein and Einstein-Maxwell fields of Weyl's type

Einsteinova a Einsteinova-Maxwellova pole Weylova typu

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Abstract

The problems solved in this thesis can be divided into two spheres. The first one represents the generation of new Einstein-Maxwell fields, the second one their physical interpretation.

The solution of non-linear, self-consistent Einstein-Maxwell equations is an enormously complicated task and was explicitly carried out only in the simplest cases. On the other hand, many exact solutions were discovered by means of special generation techniques that enable us to construct new Einstein-Maxwell fields from those already known or even from the vacuum spacetimes which do not contain any electromagnetic field. A brief review of those techniques together with a summary of necessary mathematical apparatus is given in chapter 1. It is to stress that such techniques are enormously useful, because the more exact solutions will be known and explored, the better we will be able to understand the mathematical and physical background of the theory. Thus we can precise our knowledge of the part the electromagnetic field takes in general relativity.

Chapter 2 deals with a particular generation technique used throughout the thesis and called "Horský-Mitskievitch conjecture" [7, 22]. Its principles, mathematical background and application are illustrated by three particular examples. A brief summary of results achieved by other colleagues and some considerations about an inverse problem have also been included.

The rest of the thesis is devoted to a systematic application of the Horský-Mitskievitch generating conjecture to some Weyl metrics and several Einstein-Maxwell fields are obtained in this way. Most of the generated spacetimes are really original in the context in which they are introduced, i.e. they are new as exterior Einstein-Maxwell fields of charged, finite, semi-infinite or infinite linear sources or as exterior fields of such sources inserted into a magnetic field, when they are described in Weyl cylindrical coordinates. It is to admit that a general equivalence problem, outlined in section 3.13, has not been solved because of its complexity and lack of suitable software equipment. Thus, the existence of a coordinate transformation turning the generated solutions into some already known ones cannot be excluded principally.

In chapter 3 the generating conjecture is applied to vacuum Levi-Civita spacetime and each of its Killing vectors is proved to generate an exact Einstein-Maxwell field. The solutions are classified according to Petrov types and the existence of singularities is briefly commented. The results has been accepted for publication in *Czechoslovak Journal of Physics* [48]. It is shown that in retrospection one can trace analogous steps in the generation method and propose a generalizing algorithmic scheme which is of advantage even in more complicated cases solved in chapter 4.

The problems connected with the interpretation of obtained spacetimes have also been treated. The features of radial geodesic are commented, the Newtonian gravitational potentials are found for the Weyl metrics and the conformal structure of the solutions is documented through Penrose diagrams. The discussion is illustrated by several figures in which the geodesic motion as well as the motion in the fields of corresponding Newtonian

potentials are numerically simulated. Qualitative agreement of geodesics and Newtonian trajectories is quite satisfactory. It is argued that the behaviour of spacetimes supports the interpretation presented by Bonnor [6]. Possible reinterpretation of a Bonnor-Melvin universe is proposed as a by-product.

Chapter 4 then generalizes the results obtained in chapter 4. Using coordinate transformations, we are able to interpret some of the obtained Einstein-Maxwell fields in another context and even more, we are able to apply the generating conjecture, especially the algorithmic scheme proposed in chapter 3, to other Weyl metrics, which provides some new, more general Einstein-Maxwell fields. It is proved, that some of the generated spacetimes can be understood as particular limits of the other, more general ones and finally, the obtained solutions are systematized through a limiting diagram. Therefore it may be concluded that the application of the Horský-Mitskievitch conjecture to Weyl metrics provides two complete subclasses of Einstein-Maxwell fields: one is of the electric type and the other is of the magnetic type.

The last chapter reviews the obtained results and outlines the most promising possibilities for further research in future.

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Contents

Conventions, notation and abbreviations

Throughout the text the "Landau-Lifshitz Spacelike Convention" is followed in accord with the widely recognized book by Misner, Thorne and Wheeler [36]. Used notation mostly agrees with [36] with the exception of tetrad formalism for which the notation of Chandrasekhar [8] is preferred. The reason is, that the Chandrasehkar's notation was implemented into the computer algebra package *GrTensorII* which was employed to check most of the obtained results (see page 75 or booklet [37] for more details). So called "geometrized units" defined e.g. in [36], p. 36 or in [54], p. 470 are systematically used in the whole theses. In these units, probably the most convenient ones for general relativity calculations, the speed of light c and Newton's gravitational constant G are both equal to unity: $c = G = 1$. The Einstein summation convention is also used throughout the text, that is, any repeated index is assumed to be summed over its range; for example

$$T^{\alpha}{}_{\beta}U^{\beta} = \sum_{\beta=0}^{3} T^{\alpha}{}_{\beta}U^{\beta}.$$

For all standard computations metric signature is supposed to be $(-,+,+,+)$ as in [36]. Only for the Petrov classification and for the evaluation of the Weyl scalars which requires Newman-Penrose formalism, the signature $(+,-,-,-)$ is used. This is probably not too inconsistent, as we can meet with similar compromise in some appreciated text-books such as [54]. The signature $(+,-,-,-)$ is also implemented in those sub-procedures of the *GrTensorII* package that use Newman-Penrose formalism. As a sufficient and compact introduction the that formalism the Chandrasehkar's book [8] can be recommended again. Hereinafter the vanishing cosmological constant is assumed, i.e. $\Lambda = 0$.

Notation

$\alpha,\ \beta,\ \mu,\ \nu, \dots$	Greek indices range over 0, 1, 2, 3 and represent spacetime coordinates, components, etc.
$i,\ j,\ k,\ l, \dots$	Latin indices range over 1, 2, 3 and represent spatial coordinates, components, etc.
$(\alpha),\ (\beta),\ (\mu),\ (\nu), \dots$	Greek indices range over 0, 1, 2, 3 and represent spacetime coordinates, components, etc. with respect to a orthonormal basis
$(i),\ (j),\ (k),\ (l), \dots$	Latin indices range over 1, 2, 3 and represent spatial coordinates, components, etc. with respect to a orthonormal basis
$\boldsymbol{A},\ \boldsymbol{g},\ \boldsymbol{F}$	Vectors, 1-forms, tensors in abstract notation as a multilinear mappings
$A_{\mu},\ g_{\alpha\beta},\ F_{ij}, \dots$	Covariant components of tensors
$A^{\mu},\ g^{\alpha\beta},\ F^{ij}, \dots$	Contravariant components of tensors

$A_{(\mu)},\ g^{(\alpha)(\beta)},\ F_{(i)(j)},\dots$	Covariant and contravariant tetrad components of tensors
x^α	Spacetime coordinates
$\eta_{\alpha\beta},\ \eta_{(\alpha)(\beta)}$	Minkowski metric, diagonal matrix with elements $(-1,1,1,1)$
δ^β_α	Kronecker delta tensor
$\varepsilon_{\alpha\beta\mu\nu}$	The totally antisymmetric tensor with the choice $\varepsilon_{0123}=1$
$\boldsymbol{e}_\alpha$	Basis vector of a coordinate basis, i.e. $\boldsymbol{e}_\alpha = \dfrac{\partial}{\partial x^\alpha}$
$\boldsymbol{e}_{(\alpha)},\ e_{(\alpha)\,\beta},\ e_{(\alpha)}^{\ \ \ \beta}$	Basis vector of a orthonormal tetrad basis, its covariant and contravariant components; $e_{(\alpha)}^{\ \ \ \mu} e_{(\beta)\,\mu} = \eta_{(\alpha)(\beta)}$, $\boldsymbol{e}_{(\alpha)} = e_{(\alpha)}^{\ \ \ \beta}\dfrac{\partial}{\partial x^\beta}$
$\boldsymbol{\omega}^{(\alpha)} \equiv \boldsymbol{e}^{(\alpha)}$	Basis 1-forms, dual orthonormal basis vectors; $e_{(\alpha)}^{\ \ \ \gamma} e^{(\beta)}_{\ \ \ \gamma} = \delta^{(\beta)}_{(\alpha)}$, $e_{(\gamma)}^{\ \ \ \beta} e^{(\gamma)}_{\ \ \ \alpha} = \delta^\beta_\alpha$
$\boldsymbol{*}$	Duality Hodge star operation for differential forms (see section 1.3 or e.g. [36, 54] for details)
$\boldsymbol{d}$	Exterior derivative operator
$\boldsymbol{dt},\ \boldsymbol{dr},\ \boldsymbol{d\varphi},\dots$	Coordinate basis 1-forms, gradients of the coordinate surfaces
$\otimes$	Outer product, tensor product, e.g. $(\boldsymbol{U}\otimes\boldsymbol{V})_{\alpha\beta} = U_\alpha V_\beta$
$\wedge$	Wedge product, i.e. totally antisymmetrized outer product of differential forms (see [8, 36, 54] and section 1.3)
$\boldsymbol{g},\ g_{\alpha\beta}$	Metric tensor and its components
$,$	Partial derivative: $A_{\alpha,\beta} = \dfrac{\partial A_\alpha}{\partial x^\beta}$, $A_{(\alpha),(\beta)} = \dfrac{\partial A_{(\alpha)}}{\partial x^\mu}\, e_{(\beta)}^{\ \ \ \mu}$
$\lambda_{(\alpha)(\beta)(\mu)}$	$\lambda_{(\alpha)(\beta)(\mu)} = e_{(\beta)\nu,\kappa}\left(e_{(\alpha)}^{\ \ \ \nu} e_{(\mu)}^{\ \ \ \kappa} - e_{(\alpha)}^{\ \ \ \kappa} e_{(\mu)}^{\ \ \ \nu}\right)$
$\gamma_{(\alpha)(\beta)(\mu)}$	Ricci rotation coefficients $\gamma_{(\alpha)(\beta)(\mu)} = \dfrac{1}{2}\left(\lambda_{(\alpha)(\beta)(\mu)} + \lambda_{(\mu)(\alpha)(\beta)} - \lambda_{(\beta)(\mu)(\alpha)}\right)$
$;$	Covariant derivative; in tetrad formalism that means: $T^{(\alpha)}_{\ \ \ (\beta);(\mu)} = T^{(\alpha)}_{\ \ \ (\beta),(\mu)} + \gamma^{(\alpha)}_{\ \ \ (\nu)(\mu)} T^{(\nu)}_{\ \ \ (\beta)} - \gamma^{(\nu)}_{\ \ \ (\beta)(\mu)} T^{(\alpha)}_{\ \ \ (\nu)}$
$\boldsymbol{R},\ R_{(\alpha)(\beta)(\mu)(\nu)}$	Riemann curvature tensor and its tetrad components
$R_{(\alpha)(\beta)}$	Tetrad components of the Ricci curvature tensor $R_{(\alpha)(\beta)} = R^{(\mu)}_{\ \ \ (\alpha)(\mu)(\beta)}$
R	Scalar curvature $R = R^{(\alpha)}_{\ \ \ (\alpha)}$
$\mathcal{R}$	Kretschmann scalar, full contraction of the Riemann curvature tensor
$\boldsymbol{G},\ G_{(\alpha)(\beta)}$	Einstein tensor nad its tetrad components $G_{(\alpha)(\beta)} = R_{(\alpha)(\beta)} - \dfrac{1}{2}\eta_{(\alpha)(\beta)}R$
$\boldsymbol{C},\ C_{(\alpha)(\beta)(\mu)(\nu)}$	Weyl (conformal) tensor and its tetrad components
$\Psi_0,\ \Psi_1,\ \Psi_2,\ \Psi_3,\ \Psi_4$	Weyl scalars
$\boldsymbol{T},\ T_{(\alpha)(\beta)}$	Stress energy tensor and its tetrad components
$\boldsymbol{A},\ A_{(\alpha)}$	Fourpotential of the electromagnetic field and its tetrad components

$\boldsymbol{F}$, $F_{(\alpha)(\beta)}$	Electromagnetic field tensor and its tetrad components $F_{(\alpha)(\beta)} = A_{(\beta);(\alpha)} - A_{(\alpha);(\beta)}$
$\boldsymbol{J}$, $J^{(\alpha)}$	Current density fourvector and its tetrad components
$\boldsymbol{E}$, $E^{(i)}$	Electric field strength (3-dimensional vector) and its tetrad components
$\boldsymbol{B}$, $B^{(i)}$	Magnetic field strength (3-dimensional vector) and its tetrad components

Abbreviations

Some word constructions appear so often that it seems reasonable to use folloving abbreviations:

B-M	Bonnor-Melvin
E-M	Einstein-Maxwell
H-M	Horský-Mitskievitch
L-C	Levi-Civita

Chapter 1

Introduction

The exact solutions of self-consistent Einstein-Maxwell equations represents generally an extremely complicated problem. Although some approximate methods are available, we have as yet no prescription for writing down a class of exact solution that might represent the physical situation we intend to describe. Since it is unlikely that a similar prescription will be available in close future, our efforts to find as much new exact solutions as possible might be very useful. A rich, numerous sample set of exact solutions enables us to achieve at least some piece of physical intuition and understanding of the theory. It is even more important in such non-linear problems since it is hard to get an intuition of non-linear phenomena. Thus, we can learn some important pieces of information exploring classes of exact solutions thoroughly hoping at the same time that retrospectively we might be able to trace some more general features. This is also the main aim of this thesis: to contribute to the treasury of exact solutions, to add several probably new E-M fields, to classify them algebraically and to learn how the generating conjecture formulated by Horský and Mitskievitch [22] works in all those particular cases.

1.1 Current state of the problem

Although the Einstein-Maxwell equations are very complicated to yield solutions except in a few special classes, several authors point out, they posses a large amount of hidden symmetry [29]. So far, several construction techniques have been proposed that enable us to generate new E-M fields from already known ones or even from known vacuum Einstein fields (so called seed metrics). These methods has been succesfully applied in several particular cases and some new E-M fields have been found in this way.

The striking feature of those techniques is that in some sense all of them make use of the Killing vectors describing the spacetime isometries. For many exact solutions the task to find all Killing fields need not be a trivial one [47]. If, on the contrary, one constructs new Einstein-Maxwell fields demanding at the same time that they should be endowed with some degree of symmetry, than the existence of symmetries can predictably simplify the E-M equations under some conditions. As an example of such an approach we can take E-M fields generated recently by Liang [34] and Kuang et al. [33].

Other techniques depending on the existence of the Killing vector fields were designed by Harrison [19], Geroch [25], Kinnersley [29], Cooperstock et al. [9, 10, 11] and Rácz [45, 46]. They all gradually generalized the idea of Ehlers (see e.g. [25, 29] for references) who first gave a discrete transformation mapping static vacuum solutions into stationary, not necessarily electromagnetic ones. Harrison [19] applied this method to obtain a subclass of stationary E-M fields, Geroch [25] found an explicit multi-parametric group of transformations that maps any vacuum metric admitting at least one Killing filed to a one-parameter family of new solutions of Einstein equations which "are too involved to admit any simple interpretation". The Geroch transformation is not restricted to the generation of E-M fields only, it enables also to obtain new perfect fluid solutions [45, 46]. Kinnersley [29] then used Geroch's result studying the symmetries of E-M equations in the presence of one timelike Killing vector and obtaining a five-parameter family of stationary E-M equations. Cooperstock and Cruz [9] than applied the Kinnersley's transformation to some axially symmetric vacuum metric, studied asymptotic behaviour of the obtained E-M fields and generalized the Kinnersley technique in the sense that complex coordinate transformations can be also considered. Later Carminati and Cooperstock [10, 11] enriched their approach with a technique of "coordinate modeling". They introduce special coordinates the coordinate surfaces of which coincide with the surfaces of constant electrostatic potential and they obtained some new static axially symmetric electrovac solutions in this way.

Other methods, based on the spacetime isometries and the formalism of Ernst potentials, are described in [32] where also supporting theorems and examples of their successful applications are reviewed (sections § 30.4 and § 30.5).

Hereinafter we concentrate on the generation method proposed by Horský nad Mitskievitch [22] and further generalized by Cataldo et al. [7]. The conjecture about a connection between isometries of a vacuum spacetime and the existence of the corresponding E-M field is explained in chapter 2 and is systematically applied to some vacuum Weyl metrics in chapters 3 and 4. Using H-M conjecture one can readily predict the structure of electromagnetic fourpotentials and consequently the type of the electromagnetic field he is looking for. The fact that any Killing vector in a vacuum spacetime can serve as a vector potential for some test electromagnetic field was probably first pointed out by Wald [53]. However, the difference between his approach and the H-M conjecture is principal: Wald did not solve self-consistent E-M equations looking for a new exact E-M field, he constructs a test magnetic field in the gravitational field of a Kerr black hole.

In the thesis we do not concentrate on the crucial theoretical problems connected with the H-M conjecture and its physical-geometrical roots. In several particular cases we prove that the conjecture really is a fruitful and effective tool for the generation of new E-M fields. Of course, the construction of E-M fields itself represents only one part of the problem. The second one, probably even more important from the point of physicist's view, is the interpretation of generated fields. For this task the H-M conjecture is superior to the other techniques as it enables us to inherit the physical interpretation of the seed vacuum spacetime if it is known, of course. Another advantage for the physical interpretation is that the H-M conjecture introduces only one more metric parameter characterising the strength of the electromagnetic field, while the solutions obtained through the techniques of Geroch [25] and Kinnersley [29] include several parameters the physical interpretation of which is rather ambiguous. Particularly, for the Weyl and Weyl-like metrics studied hereinafter thesis we could lean on considerable and inspiring results achieved by Bonnor [6]. Before proceeding to the explanation of the H-M conjecture and the way it will be used throughout the thesis, which can be found in chapter 2, let us briefly summarize the most important tensor definitions and key equations necessary for our calculations.

1.2 Definitions of some important tensors and invariants

The aim of this section is to summarize the analytic expressions for the terad components of tensors most often used in calculation throughout the thesis. Analogous formulae can be certainly found in any introductory textbooks of general relativity but they need not be consistent with the used sign convention (see page vii). To avoid this ambiguity the most important definitions are summarized so that the reader could possibly check the results presented in the following chapters. As throughout the thesis the tetrad formalism is used, only tetrad components are introduced. Let us remind that any tensor component or equation obtained in the coordinate basis can be easily transformed into the tetrad formalism according to the simple rules [8]

$$
\begin{aligned}
A_{(\alpha)} &= e_{(\alpha)}^{\ \ \beta} A_{\beta}, \\
A^{(\alpha)} = \eta^{(\alpha)(\mu)} A_{(\mu)} &= e^{(\alpha)}_{\ \ \beta} A^{\beta} = e^{(\alpha)\beta} A_{\beta} \\
T_{(\alpha)(\beta)} &= e_{(\alpha)}^{\ \ \mu} e_{(\beta)}^{\ \ \nu} T_{\mu\nu} \\
&\ldots
\end{aligned}
$$

On the contrary, results written in this text in a tetrad formalism can be expressed in a coordinate basis through transformations

$$
\begin{aligned}
A_{\alpha} &= e^{(\beta)}_{\ \ \alpha} A_{(\beta)}, \\
A^{\alpha} &= e_{(\beta)}^{\ \ \alpha} A^{(\beta)} = e^{(\beta)\alpha} A_{(\beta)} \\
T_{\alpha\beta} &= e^{(\mu)}_{\ \ \alpha} e^{(\nu)}_{\ \ \beta} T_{(\mu)(\nu)} \\
&\ldots
\end{aligned}
$$

All the formulae listed below are systematically derived in [8] where relevant geometrical concepts are also introduced in a comprehensive and compact way.

Components of the Riemann curvature tensor read as

$$
\begin{aligned}
R_{(\alpha)(\beta)(\mu)(\nu)} = & -\gamma_{(\alpha)(\beta)(\mu),(\nu)} + \gamma_{(\alpha)(\beta)(\nu),(\mu)} + \gamma_{(\beta)(\alpha)(\lambda)} \left(\gamma_{(\mu)}^{\ \ \ (\lambda)}{}_{(\nu)} - \gamma_{(\nu)}^{\ \ \ (\lambda)}{}_{(\mu)} \right) + \\
& + \gamma_{(\lambda)(\alpha)(\mu)} \gamma_{(\beta)}^{\ \ \ (\lambda)}{}_{(\nu)} - \gamma_{(\lambda)(\alpha)(\nu)} \gamma_{(\beta)}^{\ \ \ (\lambda)}{}_{(\mu)}.
\end{aligned}
$$

The Kretschmann scalar representing the full contraction of a Riemann tensor

$$
\mathcal{R} = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = R_{(\alpha)(\beta)(\mu)(\nu)} R^{(\alpha)(\beta)(\mu)(\nu)}.
$$

will be computed for all the spacetimes discussed in the text. Its divergence serves as a necessary, but not sufficient criterion for the existence of spacetime singularities. Performing the Petrov classification in the way described in [32] one needs the tetrad components of the Weyl (conformal) tensor

$$
\begin{aligned}
C_{(\alpha)(\beta)(\mu)(\nu)} = \ & R_{(\alpha)(\beta)(\mu)(\nu)} - \frac{1}{2} \left(\eta_{(\alpha)(\mu)} R_{(\beta)(\nu)} - \eta_{(\alpha)(\nu)} R_{(\beta)(\mu)} - \eta_{(\beta)(\mu)} R_{(\alpha)(\nu)} + \eta_{(\beta)(\nu)} R_{(\alpha)(\mu)} \right) + \\
& + \frac{1}{6} \left(\eta_{(\alpha)(\mu)} \eta_{(\beta)(\nu)} - \eta_{(\alpha)(\nu)} \eta_{(\beta)(\mu)} \right) R.
\end{aligned}
$$

and Weyl scalars expressed in some complex isotropic (Newman-Penrose) tetrad basis

$$\begin{aligned}\Psi_0 &= -\,C_{(1)(3)(1)(3)},\\ \Psi_1 &= -\,C_{(1)(2)(1)(3)},\\ \Psi_2 &= -\,C_{(1)(3)(4)(3)},\\ \Psi_3 &= -\,C_{(1)(2)(4)(2)},\\ \Psi_4 &= -\,C_{(2)(4)(2)(4)}.\end{aligned}$$

The key feature of any E-M field is the presence of an electromagnetic field, which is described either by its four-potential $\boldsymbol{A}$ or by the electromagnetic field tensor $\boldsymbol{F} = \boldsymbol{dA}$ with components $F_{(\alpha)(\beta)} = A_{(\beta);(\alpha)} - A_{(\alpha);(\beta)}$. Non-diagonal (i.e. non-zero) components of the tensor $\boldsymbol{F}$ then determine the components of two 3-dimensional spacelike vectors, the vector of the electric field strength $\boldsymbol{E}$ and the vector of the magnetic field strength $\boldsymbol{B}$ in the following way [35, 36]

$$F^{(\alpha)(\beta)} = \begin{pmatrix} 0 & E^{(1)} & E^{(2)} & E^{(3)} \\ -E^{(1)} & 0 & B^{(3)} & -B^{(2)} \\ -E^{(2)} & -B^{(3)} & 0 & B^{(1)} \\ -E^{(3)} & B^{(2)} & -B^{(1)} & 0 \end{pmatrix}.$$

The presence of the electromagnetic field curves the studied spacetimes, which is described by the traceless electromagnetic stress-energy tensor

$${T_{\rm elmg}}^{(\alpha)(\beta)} = \frac{1}{4\pi}\left(F^{(\alpha)(\mu)}F^{(\beta)}{}_{(\mu)} - \frac{1}{4}\,\eta^{(\alpha)(\beta)}F^{(\mu)(\nu)}F_{(\mu)(\nu)}\right).$$

1.3 Important equations

The results presented in the thesis arose as a solution of some important differential equations which are well-known and can be found in almost any textbook on general relativity (let us cite [8, 36, 54] as examples). This brief summary serves for reader's reference and sets definitely in what analytic form the equations are used for computations.

In the used sign convention the Einstein equations read as

$$G_{(\alpha)(\beta)} = 8\pi T_{(\alpha)(\beta)}. \tag{1.1}$$

Looking for the E-M fields we supply only the electromagnetic stress-energy tensor $\boldsymbol{T}_{\rm elmg}$ into the right hand side obtaining

$$G_{(\alpha)(\beta)} = 8\pi {T_{\rm elmg}}_{(\alpha)(\beta)}. \tag{1.2}$$

As was pointed above the tensor $\boldsymbol{T}_{\rm elmg}$ is traceless, thus we have to fulfil the equation $G^{(\alpha)}{}_{(\alpha)} = 0$ for any "pure electromagnetic" E-M field, i.e. for spacetimes curved only by the presence of the electromagnetic field.

Studying E-M fields one certainly has to work with Maxwell equations too. They are often given in an elegant, coordinate-independent language of differential forms. Then first pair of Maxwell equations reads as [35, 36]

$$\boldsymbol{ddA} = 0 \quad \text{or alternatively} \quad \boldsymbol{dF} = 0, \tag{1.3}$$

and the second pair as

$$\boldsymbol{{*}d{*}dA} = \boldsymbol{J} \quad \text{or} \quad \boldsymbol{{*}d{*}F} = \boldsymbol{J},$$

where $\boldsymbol{J}$ is a current density four-vector. While the formalism of differential forms is most effective for building the general theory of the H-M conjecture (compare analytic expressions of Eqs. (2.1) and (2.2) written both in the language of differential forms and in the coordinate approach), all practical computation are performed in some chosen coordinate frame. As the correspondence between the abstract notation and formulae for tensor components is perhaps not obvious at a first glance, let us outline, how it can be demonstrated.

For the electromagnetic field tensor [35, 36]

$$\boldsymbol{F} = F_{\mu\nu}\,\boldsymbol{dx}^\mu \otimes \boldsymbol{dx}^\nu = \frac{1}{2}\,F_{\mu\nu}\,\boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu.$$

we construct its dual 2-form [35, 36]

$$\boldsymbol{*F} = \ {*F}_{\alpha\beta}\,\boldsymbol{dx}^\alpha \otimes \boldsymbol{dx}^\beta = \frac{1}{2}\,\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}\boldsymbol{dx}^\alpha \otimes \boldsymbol{dx}^\beta = \frac{1}{4}\,\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}\,\boldsymbol{dx}^\alpha \wedge \boldsymbol{dx}^\beta.$$

The exterior derivative then leads to a 3-form

$$\boldsymbol{d*F} = \ {*F}_{\alpha\beta;\lambda}\,\boldsymbol{dx}^\lambda \wedge \boldsymbol{dx}^\alpha \wedge \boldsymbol{dx}^\beta = \frac{1}{4}\,\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}{}_{;\lambda}\,\boldsymbol{dx}^\lambda \wedge \boldsymbol{dx}^\alpha \wedge \boldsymbol{dx}^\beta$$

with coordinates

$$(\boldsymbol{d*F})_{\lambda\alpha\beta} = \frac{3!}{4}\,\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}{}_{;\lambda}.$$

Next, according to the second pair of Maxwell equations there exists another 3-form $\boldsymbol{*J}$ such that

$$\boldsymbol{d*F} = \ 4\pi\,\boldsymbol{*J} = 4\pi\,{*J}_{\lambda\alpha\beta}\,\boldsymbol{dx}^\lambda \otimes \boldsymbol{dx}^\alpha \otimes \boldsymbol{dx}^\beta.$$

Now it is possible to construct a dual object $\boldsymbol{*}(\boldsymbol{*J}) = \boldsymbol{J}$ representing a 1-form $\boldsymbol{J}$ the contravariant components of which read as [36]

$$4\pi J^\kappa = (\boldsymbol{*d*F})^\kappa = 4\pi\frac{1}{3!}\,{*J}_{\lambda\alpha\beta}\varepsilon^{\lambda\alpha\beta\kappa} = \frac{1}{3!}\,(\boldsymbol{d*F})_{\lambda\alpha\beta}\,\varepsilon^{\lambda\alpha\beta\kappa} = \frac{1}{4}\,\varepsilon_{\alpha\beta\mu\nu}\varepsilon^{\lambda\alpha\beta\kappa}F^{\mu\nu}{}_{;\lambda}.$$

Substituting the identity [35, 36]

$$\varepsilon_{\alpha\beta\mu\nu}\varepsilon^{\lambda\alpha\beta\kappa} = \varepsilon_{\alpha\beta\mu\nu}\varepsilon^{\alpha\beta\lambda\kappa} = -2\,\delta^{\lambda\kappa}{}_{\mu\nu} = 2\left(\delta^\lambda{}_\nu\delta^\kappa{}_\mu - \delta^\lambda{}_\mu\delta^\kappa{}_\nu\right)$$

and taking into account the antisymmetry of the electromagnetic field tensor we finally come to

$$4\pi J^\kappa = (\boldsymbol{*d*F})^\kappa = F^{\kappa\mu}{}_{;\mu}.$$

All spacetimes described below in this text are interpreted as exterior fields of mass sources. In such case we set $\boldsymbol{J} = 0$ which provides sourceless Maxwell equations. The second pair of them then takes the form [35, 36]

$$\boldsymbol{*d*dA} = 0 \quad \text{or} \quad \boldsymbol{*d*F} = 0. \tag{1.4}$$

As in the text the tetrad formalism is mainly used, let us complete this part with the sourceless Maxwell equations written in tetrad formalism. First pair $\boldsymbol{dF} = 0$ can be written as

$$F_{(\alpha)(\beta);(\mu)} + F_{(\beta)(\gamma);(\alpha)} + F_{(\mu)(\alpha);(\beta)} = 0. \tag{1.5}$$

and the second pair $\boldsymbol{*d*F} = 0$ as

$$F^{(\alpha)(\beta)}{}_{;(\beta)} = 0. \tag{1.6}$$

The existence of Killing vectors is very important for the method through which we have obtained new E-M fields (see chapter 2). Let us remind that any Killing vector $\boldsymbol{\xi}$ has to satisfy so called Killing equation [22, 35, 54]

$$\xi_{(\alpha);(\beta)} + \xi_{(\beta);(\alpha)} = 0. \tag{1.7}$$

Although the spacetimes presented in the following chapters represent solutions both of the Maxwell equations and of the Einstein ones, we do not reproduce the analytic form of those equations in all particular cases. The reason is obvious: (i) reproducing all the differential equations would make the thesis unbearably lengthy and (ii) often it is much more comfortable to check the result against the equations with a suitable computer algebra software. It should be admitted that in case of spacetimes (4.16), (4.18), (4.25), (4.26), (4.34), (4.36) the E-M equations were not solved at all; the metric and vector potential were "guessed" according to the scheme formulated in section 3.2 and then it was verified that they really represent a solution of E-M equations. That is also why further in the text we refer to the formulae summarized above only verbally.

Chapter 2

Generating conjecture

This chapter is devoted to the principles of the generating conjecture formulated by Horský and Mitskievitch [22] and further generalized by Cataldo et al. [7]. In section 2.1 key ideas of the generalized H-M conjecture are resumed, section 2.2 includes a brief survey of results obtained through employing the conjecture to various vacuum spacetimes. Attention is concentrated on the cases that have not been discussed by other authors in this context.

2.1 The H-M conjecture and its application

The H-M conjecture proposed in [22] outlines an efficient and fruitful way, how to obtain solutions of E-M equations as a generalization of some already known vacuum seed metrics. Its mathematical background is based on the striking analogy between equations satisfied by Killing vectors $\boldsymbol{\xi}$ in *vacuum* spacetimes ([7, 35, 53, 54])

$$\boldsymbol{*d*d\xi} = 0 \quad \equiv \quad \xi^{(\alpha);(\beta)}{}_{;(\beta)} + \underbrace{R^{(\alpha)}{}_{(\mu)}}_{=0} \xi^{(\mu)} = \xi^{(\alpha);(\beta)}{}_{;(\beta)} = 0 \tag{2.1}$$

and vacuum (sourceless) Maxwell equations for a testing electromagnetic four-potential $\boldsymbol{A}$

$$\boldsymbol{*d*dA} = 0 \quad \equiv \quad F^{(\alpha)(\beta)}{}_{;(\beta)} = -A^{(\alpha);(\beta)}{}_{;(\beta)} + A^{(\beta);(\alpha)}{}_{;(\beta)} = 0. \tag{2.2}$$

This connection is well known for a long time and is pointed out e.g. in [35, 53, 54].

This suggestive coincidence inspired Horský and Mitskievitch in [22] to formulate the conjecture that can be expressed in the following way (quoted verbally according to [7]):

"The electromagnetic four-potential of a stationary self consistent Einstein-Maxwell field is simultaneously proportional (up to constant factor) to the Killing covector of the corresponding vacuum spacetime when the parameter connected with the electromagnetic field of the self-consistent problem is set equal to zero, this parameter coinciding with the afore-mentioned constant factor."

Let $\boldsymbol{g}$ denotes the metric tensor of a vacuum seed metric, q parameter characterizing the strength of the electromagnetic field (mentioned in the quotation above) and $\tilde{\boldsymbol{g}} = \tilde{\boldsymbol{g}}(q)$ the metric tensor of an E-M field, representing in fact one-parameter class of solutions. According to the conditions of the H-M conjecture

$$\lim_{q \to 0} \tilde{\boldsymbol{g}} = \boldsymbol{g};$$

i.e. in the case of null electromagnetic field $q = 0$ one comes back to the original seed vacuum metric. The above quoted formulation of the conjecture was further generalized by

Cataldo, Kunaradtya and Mitskievitch [7] so that the four-potential $\boldsymbol{A}$ need not inevitably equal to a Killing vector of the seed solution multiplied only by a constant factor, but can also represent this Killing vector multiplied by a suitable scalar function $\mathcal{F}$. The function is evidently not arbitrary; it must satisfy sourceless Maxwell equations

$$\boldsymbol{{*}d{*}d}(\mathcal{F}\boldsymbol{\xi}) = 0$$

with respect to $\tilde{\boldsymbol{g}}$ (see [7]). Here the constant parameter q must be involved in the analytical expression of the function $\mathcal{F}$. From the point of this generalization it is not necessary to demand that $\boldsymbol{\xi}$, a Killing vector with respect to $\boldsymbol{g}$, must be also a Killing vector with respect to $\tilde{\boldsymbol{g}}$ as used to be argued (see e.g. [39]). This generalization leads to more vague connection between the Killing vectors and four-potentials. At the same time, however, it enables to find a wealth of situations in which the presumptions of this generalized H-M conjecture are fulfilled (see section 2.2). Although the original formulation quoted above restricts itself only to vacuum seed spacetimes $\boldsymbol{g}$ and static or stationary E-M fields $\tilde{\boldsymbol{g}}$, nowadays these conditions seem to become redundant. The possibility to construct new solutions also from non-vacuum seed metrics is supposed immediately in [22]. In section 3.7 a non-stationary solution will be obtained through the procedure of the generalized conjecture.

Probably the most important advantage of the H-M conjecture is the opportunity to choose the character of the electromagnetic field we would like to obtain. It is obviously determined by the vector potential and thus by the geometrical substance of the Killing vector one uses for the generation of $\tilde{\boldsymbol{g}}$. If the seed metric $\boldsymbol{g}$ admits more than one Killing vector, there is usually possible to construct more E-M fields, each of them corresponding to a different Killing vector. The situation is more lucid when the seed metric $\boldsymbol{g}$ is static. In that case a straightforward calculation leads to the conclusion that rotational, as well as space-like translational Killing vectors, give magnetic E-M fields, while timelike translational and the boost Killing vectors lead to the electric fields (in full analogy with the Minkowski spacetime). For the rotational Killing vector ∂_φ in common cylindrical coordinates the correspondence with longitudinal magnetic field was demonstrated by Wald (see e.g. [35], p. 66, 326). Let us remind that the electric or magnetic character of any obtained E-M field determines the sign of the electromagnetic invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = 2\left(B^2 - E^2\right), \tag{2.3}$$

where $\boldsymbol{F}$ is an antisymmetric tensor of the electromagnetic field related to the components of electric and magnetic field strengths $\boldsymbol{E}$ and $\boldsymbol{B}$ in a standard way (see e.g. [35], p. 23, [36], p. 74). If $F_{(\mu)(\nu)}F^{(\mu)(\nu)} > 0$, the field is of magnetic type, if $F_{(\mu)(\nu)}F^{(\mu)(\nu)} < 0$ then it is of electric type.

The non-existence of a general algorithm makes the practical application of the conjecture rather difficult. Firstly, there is no general guaranty that the calculations should lead to solvable system of equations and secondly, we do not know in what way the metric $\tilde{\boldsymbol{g}}$ should differ from $\boldsymbol{g}$. Cataldo's generalized formulation then brings another uncertainty into the choice of the vector potential $\boldsymbol{A}$. Thus generating new Einstein-Maxwell fields requires some intuition and usually entails several trials and impasses. Fortunately, for all the cases described in chapters 3 and 4, there was possible to formulate an algorithmic scheme (see section 3.2) the following of which leads to the desired result.

From the theoretical point of view the H-M conjecture still represents an open geometrical-physical problem. We have no general proof at our disposal and we do not know exact limits of its applicability. That means we still do not understand its physical background properly. Therefore, in next chapters these crucially important questions are rather pragmatically put aside and considerations concentrate only on the practical mechanism of generating new E-M fields. Particularly, next chapter 3 is devoted to the application of the conjecture to the Levi-Civita vacuum spacetime and chapter 4 then further generalizes obtained results.

2.2 Generating conjecture and already known metrics

This section illustrates the key points of H-M conjecture in some concrete situations fulfilling all conditions of Cataldo's formulation [7]. The aim is to provide a survey of so far studied metrics and outline possible limits. It should be stressed that all spacetimes below were found in another way, even before the H-M conjecture itself was formulated. The number of particular examples in which the H-M conjecture can be employed is not negligible. This fact seems to exclude the possibility that the correspondence between Killing vectors and four-potentials might be mere interplay of coincidences.

Several important examples were already mentioned by Horský and Mitskievitch [22], in [7, 23, 52] the application of conjecture leads to new E-M fields. Explored spacetimes known to the author of this work are summarized in the following table:

E-M spacetime	*Corresponding seed metric*	*References*
Reissner-Nordström	Schwarzchild	[22]
Kerr-Newmann	Kerr or Minkowski	[22, 24]
McVittie	Taub	[18, 22, 23, 39]
Bonnor-Melvin	Minkowski	[23]
Generalized Kasner	Kasner	[23]
McCrea	Van Stockum	[51, 52]
Chitre et al.	one static vacuum metric	[51, 52]
Van den Berg and Wils	one static vacuum metric	[51, 52]
Pencil of light in a Bonnor-Melvin Universe	Pencil of light	[7]

The following examples have not been discussed in connection with the H-M conjecture yet. Therefore we are going to deal with them in more details.

2.2.1 Datta's spacetime

The non-stationary E-M field described by the line element

$$ds^2 = -\frac{1}{A(t)}\,\mathrm{d}t^2 + A(t)\,\mathrm{d}x^2 + B(t)\,\mathrm{d}y^2 + C(t)\,\mathrm{d}z^2, \tag{2.4}$$

where

$$B(t) = C(t) = t^2, \quad A(t) = \frac{b}{t} - \frac{q^2}{t^2}; \quad q > 0$$

was found by Datta in 1965, basic characteristic are resumed also in Kramer et al. [32], § 11.3.3 Eq. (11.60) where the metric is mentioned in a broader context of spacetimes with the same groups of symmetry (isometry group G_4 with isotropy subgroups of 1-dimensional spatial rotations and 2-dimensional isotropy of of non-null electromagnetic field consisting from boosts and spatial rotations). Setting $q = 0$ one obviously gets a vacuum equation. For further calculations we employ the tetrad

$$\boldsymbol{\omega}^{(0)} = \frac{1}{\sqrt{A(t)}}\,\boldsymbol{dt}, \quad \boldsymbol{\omega}^{(1)} = \sqrt{A(t)}\,\boldsymbol{dx}, \quad \boldsymbol{\omega}^{(2)} = \sqrt{B(t)}\,\boldsymbol{dy}, \quad \boldsymbol{\omega}^{(3)} = \sqrt{C(t)}\,\boldsymbol{dz}.$$

The four-potential

$$\boldsymbol{A} = \frac{q}{t}\,\boldsymbol{dx} = \frac{q}{\sqrt{bt - q^2}}\,\boldsymbol{\omega}^{(1)}$$

is collinear with the Killing vector ∂_x of the corresponding vacuum metric and determines an electromagnetic field of an electric type with the only non-zero tetrad component of electromagnetic tensor $\boldsymbol{F}$

$$F^{(0)(1)} = E^{(1)} = -\,\frac{q}{t^2}$$

and the electromagnetic invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = -2\,\frac{q^2}{t^4}.$$

Computing the Einstein tensor, Kretschmann and Weyl scalars we obtain

$$G_{(0)(0)} = -\,G_{(1)(1)} = G_{(2)(2)} = G_{(3)(3)} = \frac{q^2}{t^4}, \quad \text{other} \quad G_{(\mu)(\nu)} = 0,$$

$$\mathcal{R} = 4\,\frac{3b^2t^2 - 12q^2bt + 14q^4}{t^8},$$

$$\Psi_2 = \frac{2q^2 - bt}{2t^4}$$

Therefore the metric belongs to the Petrov class D.

2.2.2 The Kowalczyński and Plebański metric

This E-M field discovered in 1977 is also included in [32], § 27.7, Eq. (27.57). In more details it is studied in the original source [31], various useful and interesting background information concerning not only this class of E-M spacetimes are presented also in papers [30, 43]. The line element can be written in the form

$$\mathrm{d}s^2 = -2\,\frac{B(z)\,\mathrm{d}t^2}{x^2} + 2\,\frac{\mathrm{d}x^2}{x^2\,A(x)} + 2\,\frac{A(x)\,\mathrm{d}y^2}{x^2} + 2\,\frac{\mathrm{d}z^2}{x^2\,B(z)}, \tag{2.5}$$

where

$$A(x) = -2\,q^4c^4x^4 + cx^3 - ax^2, \quad B(z) = az^2 + b,$$

a, b, c, d being real constants. Choosing the tetrad

$$\boldsymbol{\omega}^{(0)} = \frac{\sqrt{2B(z)}}{x}\,\boldsymbol{dt}, \quad \boldsymbol{\omega}^{(1)} = \frac{\sqrt{2}}{x\sqrt{A(x)}}\,\boldsymbol{dx}, \quad \boldsymbol{\omega}^{(2)} = \frac{\sqrt{2A(x)}}{x}\,\boldsymbol{dy}, \quad \boldsymbol{\omega}^{(3)} = \frac{\sqrt{2}}{x\sqrt{B(z)}}\,\boldsymbol{dz},$$

and taking the four-potential

$$\boldsymbol{A} = 2q\,(\,cx - a\,)\,\boldsymbol{dy} = \frac{\sqrt{2}\,qx\,(\,cx - a\,)}{\sqrt{A(x)}}\,\boldsymbol{\omega}^{(2)}$$

after standard calculations we obtain the electromagnetic field of magnetic type with the component of electromagnetic field tensor

$$F^{(1)(2)} = B^{(3)} = qcx^2$$

and electromagnetic invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = 2\,q^2c^2x^4.$$

Looking for the Einstein tensor one gets non-zero components

$$G_{(0)(0)} = G_{(1)(1)} = G_{(2)(2)} = -\,G_{(3)(3)} = q^2c^2x^4.$$

Finally, the Kretschmann scalar

$$\mathcal{R} = c^2x^6\left(56\,q^4x^2c^2 - 24\,q^2cx + 3\right),$$

and the only non-zero Weyl scalar

$$\Psi_2 = -\,\frac{1}{4}\,x^3\,\left(4\,q^2x - c\right)$$

show that the metric belongs to the Petrov type D. Setting $q = 0$ we again come to an vacuum solution of Einstein equations.

2.2.3 Conformally flat E-M field

The line-element

$$\mathrm{d}s^2 = -(1+q^2z^2)\,\mathrm{d}t^2 + (1-q^2y^2)\,\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{1-q^2y^2} + \frac{\mathrm{d}z^2}{1+q^2z^2} \tag{2.6}$$

represents the only conformally flat, non-isotropic, sourceless E-M field. Basic facts are extracted in [32], § 10.3, Eqs. (10.19) and (10.20). Working with the tetrad

$$\begin{aligned}
\boldsymbol{\omega}^{(0)} &= \sqrt{1+q^2z^2}\,\boldsymbol{dt}, & \boldsymbol{\omega}^{(1)} &= \sqrt{1-q^2y^2}\,\boldsymbol{dx},\\
\boldsymbol{\omega}^{(2)} &= \frac{1}{\sqrt{1-q^2y^2}}\,\boldsymbol{dy}, & \boldsymbol{\omega}^{(3)} &= \frac{1}{\sqrt{1+q^2z^2}}\,\boldsymbol{dz},
\end{aligned}$$

and taking the four-potential

$$\boldsymbol{A} = qz\cos\beta\,\boldsymbol{dt} + qy\sin\beta\,\boldsymbol{dx} = \frac{qz\cos\beta}{\sqrt{1+q^2z^2}}\,\boldsymbol{\omega}^{(0)} + \frac{qy\sin\beta}{\sqrt{1-q^2y^2}}\,\boldsymbol{\omega}^{(1)}, \quad \beta = \text{const.}$$

we easily check, that the electromagnetic field has generally both electric and magnetic components

$$F^{(0)(3)} = E^{(3)} = -\,q\cos\beta, \qquad F^{(1)(2)} = B^{(3)} = -\,q\sin\beta,$$

which is also seen from the form of the invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = -\,2q^2\cos\left(2\beta\right). \tag{2.7}$$

The dependence of $F_{(\mu)(\nu)}F^{(\mu)(\nu)}$ on β and q is sketched in Fig. 2.1 and demonstrates quite an interesting feature of this E-M field: the type of the electromagnetic field is

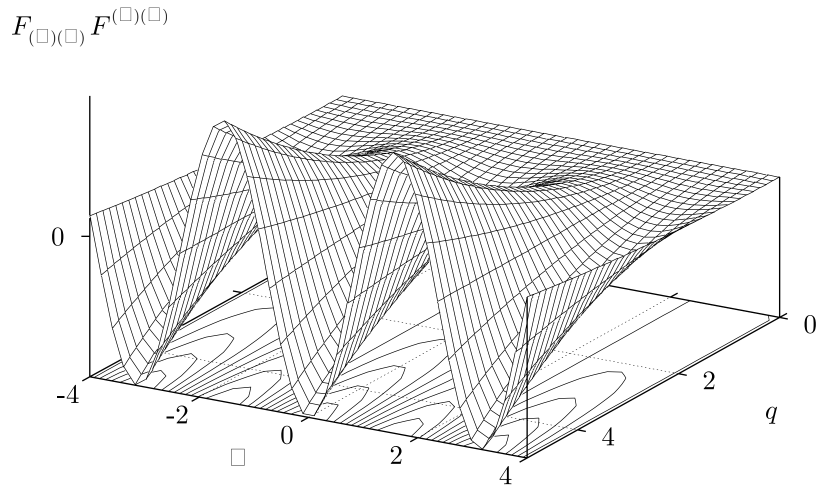


Fig. 2.1: The dependence of $F_{(\mu)(\nu)}F^{(\mu)(\nu)}$ on β and q.

fully determined by the parameter β that does not enter the expression (2.6) for the line element. Thus the same metric (2.6) can describe E-M field of either electric or magnetic type. It should be noted, however, that both electric and magnetic field are collinear. We shall meet an analogical situation in the chapter 3 comparing solutions described in sections 3.4 and 3.8.

The non-zero tetrad components of the Einstein tensor read as

$$G_{(0)(0)} = G_{(1)(1)} = G_{(2)(2)} = -\,G_{(3)(3)} = q^2$$

and the Kretschmann scalar then as

$$\mathcal{R} = 8q^4.$$

Algebraic classification returns Petrov type 0. The limit $q \to 0$ obviously leads to the Minkowski spacetime.

2.3 An inverse problem

There above mentioned examples fulfil the presumptions of the H-M conjecture and therefore they could have been principally generated from their vacuum limits $q \to 0$. No matter that they were found in a different way, their connection with the H-M conjecture is also worth mentioning for another reason.

First, the H-M conjecture might help to determine the vector potential of the electromagnetic field. In many references (let us mention [32] as an example) only basic characteristic of E-M Maxwell fields are given and it is not always straightforward what four-potential corresponds to the studied electromagnetic field. In this sense the H-M conjecture considerably reduces the number of "suspicious" possibilities as it suggests to try vectors collinear to the Killing vectors of a vacuum limit of the E-M field (in fact, very often this vector is also a Killing vector in a "charged" case). With the help of a computer it is usual much more quick than to look for a resource paper especially in case of solutions discovered 10 or more years ago. This method was used also in the three examples discussed above.

The second, more difficult problem treated successfully first by Štefaník and Horský [51, 52] is quite inverse to the generating conjecture: you have an E-M field and you would like to find its vacuum counterpart. It is not always as simple as for the examples above, when setting $q = 0$ was sufficient. For some E-M fields no corresponding vacuum metric exists, so the inverse problem generally even need not have an solution. As an example we can take a metric

$$\begin{aligned} \mathrm{d}s^2 = & -f\,(\,\mathrm{d}t + \omega\,\mathrm{d}\varphi)^2 + f^{-1}a_0^2 \left[\mathrm{d}\sigma^2 - \left(\frac{1+\sigma^2}{1+u^2}\right)^2 \frac{1-1/A^2}{1+1/A^2}\,\mathrm{d}u^2 \right] - \\ & - f^{-1}(A-1/A)^3(A+1/A)\,\mathrm{d}\varphi^2 \end{aligned} \tag{2.8}$$

with

$$f = \frac{(1+\sigma^2)(1+u^2)}{(1+\sigma^2)\left[1+A^2(1+u^2)\right] + 2A\sqrt{(1+\sigma^2)(1+u^2)}},$$

$$\omega = -2\frac{A+1/A}{\sqrt{(1+\sigma^2)(1+u^2)}}\,u - 3\arctan u - \frac{u}{1+u^2}$$

described in [40]. The four-potential

$$\boldsymbol{A} = A_t\,\boldsymbol{dt} + A_\varphi\,\boldsymbol{d\varphi},$$

$$A_t = \frac{1+\sigma^2 + A\sqrt{(1+\sigma^2)(1+u^2)}}{(1+\sigma^2)\left[1+A^2(1+u^2)\right] + 2A\sqrt{(1+\sigma^2)(1+u^2)}}\,u,$$

$$A_\varphi = \omega A_t - \frac{A+1/A}{\sqrt{(1+\sigma^2)(1+u^2)}} - \frac{1}{1+u^2},$$

cannot be set equal zero (another interesting feature of this E-M field is that this field is of an magnetic type in some regions and of electric type in other ones — see [40] for details). The physical interpretation of this fact is not clear. Our intuition argues, that any physically relevant E-M should have its vacuum limit, if it was only the Minkowski spacetime. One comes to this conclusion when he realizes, that the electromagnetic field is connected either with charged particles or with an electromagnetic radiation propagating in some background spacetime. From this point of view the solvability of the problem inverse to the H-M conjecture principally could serve as a necessary (but not sufficient) condition to identify the E-M field with some realistic physical sources.

Chapter 3

Einstein-Maxwell fields of Levi-Civita's type

In this part the method outlined in [22] and in previous chapters is systematically applied to the vacuum L-C metric which generally admits 3 Killing vectors and another Killing field for two special choices of the metric parameters. It is shown that every Killing vector is connected with a new class of Einstein-Maxwell fields and each of those classes is found explicitly.

Firstly, the basic characteristics of the L-C solution are resumed and the key ideas of the H-M conjecture are adopted to the L-C seed metric. The application of the generating conjecture then gradually leads to five new classes of the E-M equations, each of which corresponds to one Killing vector of the seed L-C vacuum metric. Moreover, another class obtained as a by-product of computations is added. This latter solution is namely interesting for the reason that for special values of its parameters it reduces to the B-M universe filled not by magnetic but by electric background. The rest of the chapter is devoted to a brief discussion of the physical characteristics of the generated spacetimes; the study of radial geodesic motion is supplemented by four particular examples of radial geodesics, the key features of the corresponding Newtonian gravitational sources are illustrated by four particular trajectories and finally, Penrose conformal diagrams for both seed and generated spacetimes are drawn and qualitatively discussed. In each case the presence of at least one singularity is detected and the main physical characteristics of each of the spacetimes are always compared with the situation in the seed L-C metric. Most results presented in this chapter can be also find in [48].

3.1 The L-C solution

The line element of the L-C static vacuum spacetime can be written in the Weyl form [6, 55]

$$\mathrm{d}s^2 = -r^{4\sigma}\mathrm{d}t^2 + r^{4\sigma(2\sigma-1)}\left(\mathrm{d}r^2 + \mathrm{d}z^2\right) + C^{-2}r^{2-4\sigma}\mathrm{d}\varphi^2, \tag{3.1}$$

where $\{t, r, \varphi, z\}$ are usual cylindrical coordinates: $-\infty < t, z < \infty$, $r \geq 0$, $0 \leq \varphi < 2\pi$, the hypersurfaces $\varphi = 0$, $\varphi = 2\pi$ are identified. The expression (3.1) contains two arbitrary constants σ, C, both of them should be fixed by the internal composition of the physical source. The constant C refers to the deficit angle, and cannot be removed by scale transformations. This angular defect is often interpreted as representing gauge cosmic strings (see [55] and the bibliography therein). The physical importance of the other parameter σ is mostly understood in accordance with the Newtonian analogy of the L-C solution — the gravitational field of an infinite uniform line-mass ("infinite wire") with the linear mass density σ [6, 55]. Newtonian counterparts corresponding to some spacetimes of the Levi-Civita's type including the L-C metric are surveyed in section 3.11.

All orthonormal bases employed in the calculations below are always chosen as a generalization of the set

$$\begin{aligned} \boldsymbol{\omega}^{(0)} &= r^{2\sigma}\boldsymbol{dt}, & \boldsymbol{\omega}^{(1)} &= r^{2\sigma(2\sigma-1)}\boldsymbol{dr}, \\ \boldsymbol{\omega}^{(2)} &= C^{-1}r^{1-2\sigma}\boldsymbol{d\varphi}, & \boldsymbol{\omega}^{(3)} &= r^{2\sigma(2\sigma-1)}\boldsymbol{dz}, \end{aligned} \tag{3.2}$$

probably the simplest tetrad one can use for the L-C solution. The Kretschmann scalar

$$\mathcal{R} = R_{(\alpha)(\beta)(\mu)(\nu)}R^{(\alpha)(\beta)(\mu)(\nu)} = 64\sigma^2\left(4\sigma^2-2\sigma+1\right)(2\sigma-1)^2\, r^{-16\sigma^2+8\sigma-4}, \tag{3.3}$$

where $R_{(\alpha)(\beta)(\mu)(\nu)}$ are the components of the Riemann tensor in a chosen orthonormal basis, is infinite at $r=0$ for all σ, C excluding $\sigma=0$ and $\sigma=\frac{1}{2}$ when the spacetime is flat (see below). Thus metric (3.1) has a singularity along the z-axis $r=0$ that is preferably interpreted as the infinite line source. There is evidently no horizon, the spacetime is asymptotically flat in the radial direction for $\sigma\neq 0, 1/2$.

The analytic form of non-zero Weyl scalars

$$\begin{aligned} \Psi_0 &= \Psi_4 = (2\sigma-1)(2\sigma+1)\,\sigma r^{-8\sigma^2+4\sigma-2}, \\ \Psi_2 &= (2\sigma-1)^2\,\sigma r^{-8\sigma^2+4\sigma-2} \end{aligned}$$

leads to the conclusion that the L-C metric (3.1) belongs generally to the Petrov type I with the exception of algebraically special cases belonging either to the Petrov type 0 or to the Petrov type D:

σ	Petrov type	Weyl Scalars
$0,\ \frac{1}{2}$	0	all zero
$-\frac{1}{2}$	D	$\Psi_2 = -\frac{2}{r^6}$
1	D	$\Psi_0 = \Psi_4 = \frac{3}{r^6},\ \Psi_2 = \frac{1}{r^6}$
$\frac{1}{4}$	D	$\Psi_0 = \Psi_4 = -\frac{3}{16}r^{-3/2},\ \Psi_2 = \frac{1}{16}r^{-3/2}$

As any static cylindrically symmetric solution, the metric (3.1) admits three Killing vectors:

$$\boldsymbol{\xi}_z = r^{4\sigma(2\sigma-1)}\boldsymbol{dz} = r^{2\sigma(2\sigma-1)}\boldsymbol{\omega}^{(3)} \Longleftrightarrow \partial_z, \tag{3.4.a}$$

$$\boldsymbol{\xi}_\varphi = C^{-2}r^{2-4\sigma}\boldsymbol{d\varphi} = C^{-1}r^{1-2\sigma}\boldsymbol{\omega}^{(2)} \Longleftrightarrow \partial_\varphi, \tag{3.4.b}$$

$$\boldsymbol{\xi}_t = r^{4\sigma}\boldsymbol{dt} = r^{2\sigma}\boldsymbol{\omega}^{(0)} \Longleftrightarrow \partial_t; \tag{3.4.c}$$

these Killing vectors determine the integrals of motion for geodesic trajectories. In the case $\sigma=\frac{1}{4}$, resp. $\sigma=-\frac{1}{2}$ the L-C solution has another Killing vector $\boldsymbol{\xi}_{1/4} = -\varphi r\boldsymbol{dt} + tr\boldsymbol{d\varphi}$, resp. $\boldsymbol{\xi}_{-1/2} = -zr^4\boldsymbol{d\varphi} + \varphi r^4\boldsymbol{dz}$. The former corresponds to a Lorentz boost in the $t-\varphi$

plane, the latter to a rotation in the $\varphi - z$ plane. All mentioned Killing vectors can generate E-M fields as is shown below.

Although the solution 3.1 was found by Tulio Levi-Civita in 1919 (see e.g. references in [55]), the problem of its physical interpretation is not solved completely. As mentioned above, the "standard" approach usable for most Weyl's solutions identifies the L-C metric with the gravitational field of an infinite line-mass spread along the z-axis. The argument is based on the analytic form of the gravitational potential for the corresponding Newtonian counterpart (see section 3.11 for more details). No matter how convenient this may seem, such conclusion is unquestionably accepted only for some values of the parameter σ. As pointed out by Bonnor and Martins [5], several objection against this interpretation arises. Negative values $\sigma < 0$ lead to the negative linear density of the Newtonian analogy and thus violate the energy conditions, circular timelike geodesics exist only for $0 < \sigma < 1/4$, for $\sigma = 1/4$ they become null. Moreover, once identifying the L-C solution with an infinite line-mass source with the linear density 2σ one should expect, that when the parameter σ increases (which should mean that we add more energy to the source), the spacetime should be more curved. Unfortunately, the dependence of the Kretschmann scalar $\mathcal{R}$ given by (3.3) and shown in Fig. 3.1 is more complicated. We can see that $\mathcal{R}$ really increases in the interval $\sigma \in (0, 1/4)$, but then decreases to zero for $\sigma = 1/2$, in which case the L-C solution becomes flat. Than $\mathcal{R}$ again increases up to $\sigma \approx 1$ and for higher values of σ slowly decreases. Note, that the value $\sigma = 1$ represents $10^{28}\,\mathrm{g} \cdot \mathrm{cm}^{-1}$ [6]. Finally, in cases $\sigma = 0, \frac{1}{2}$ when the metric is flat, the parameter σ cannot be interpreted as a linear density at all.

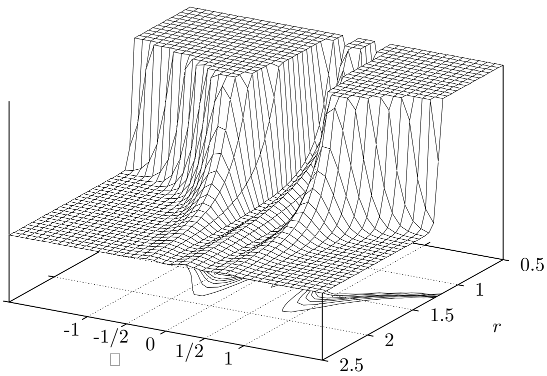


Fig. 3.1: The dependence of the Kretschmann scalar on σ and r.

Several possible sources have been suggested for which the L-C metric could represent an exterior solution [16, 42, 55]. Neither of them admits all possible values of σ. They can be divided both into cylinder and wall sources, which bears another open problem: should we regard the L-C solution as cylindrically symmetric or rather as plane symmetric? In the latter case the restriction laid on the range of the coordinate φ certainly becomes redundant. While Philibin [42] matches the L-C solution to a cylindrical source for $|\sigma| < 1/2$ and to a wall source for $|\sigma| > 1/2$, Wang et al. [55] recently constructed a cylindrical shell of an anisotropic fluid satisfying the energy conditions for the mass parameter σ in the range $0 \leq \sigma \leq 1$. Thus although the value of σ is not bound by any mathematical relation, the existence of realistic physical sources sets strict limits.

Another difficulty in the interpreting of the L-C solution was again demonstrated by Bonnor [4, 6]. In case $\sigma = -1/2, C = 1$ the metric (3.1) can be transformed either into Taub's plane symmetric solution, or into the Robinson-Trautman solution, or into the solution describing the gravitational field of a semi-infinite line-mass. Each of this possibilities suggests a different physical interpretation. It is also worth noting, that while most vacuum static Weyl solutions, including the Curson and the Darmois-Vorhees-Zipoy solutions, can arise as the metrics of counter-rotating relativistic disks [2, 3], the L-C solution does not seem to admit this interpretation; we have no such source at our disposal nowadays. More information about the L-C metric, especially about the possible character of the source, can be found in [5, 6, 16, 42, 55] and in the references cited therein.

In this text the traditional interpretation is preferred because of its simplicity and satisfying correspondence with appropriate Newtonian sources achieved in section 3.11. However, accepting Bonnor arguments [6], one can identify the L-C solution with an infinite line source only for $0 < \sigma < \frac{1}{4}$ to avoid above mentioned annoyances.

3.2 Application of the conjecture

Spacetimes treated below in this chapter were found through the H-M conjecture from the L-C seed metric. The procedure leads to six new solutions containing different types of the electromagnetic field. For the sake of efficiency all tensor components were expressed in tetrad formalism. Despite the absence of an algorithm to follow (see chapter 2), in the particular case of the L-C seed metric it was retrospectively possible to trace back the analogous steps in calculations and resume them in the following scheme:

a) According to (3.2) and (3.4) Killing vectors of $\boldsymbol{\xi}_z$, $\boldsymbol{\xi}_\varphi$, $\boldsymbol{\xi}_t$ are collinear with one of the basis vectors (3.2). It is possible to choose the vector potential $\boldsymbol{A}$ in such a way that its *tetrad* components with respect to the demanded metric $\tilde{\boldsymbol{g}}$ coincide - up to a constant factor q characterizing the strength of the electromagnetic field - with the components of the Killing vectors with respect to $\boldsymbol{g}$, i.e. with respect to (3.2). In the case of $\boldsymbol{\xi}_{1/4}$ and $\boldsymbol{\xi}_{-1/2}$ (see sections 3.7, 3.6) the tetrad vectors were chosen in such a way that one of the basis vectors coincides either with $\boldsymbol{\xi}_{1/4}$ or with $\boldsymbol{\xi}_{-1/2}$. Then, for simplicity's sake a suitable coordinate transformation was performed.

b) The question is, in what way we should modify $\tilde{\boldsymbol{g}}$ in relation to $\boldsymbol{g}$ or (in tetrad formalism), in what way the orthonormal basis used for $\tilde{\boldsymbol{g}}$ should differ from that one used for $\boldsymbol{g}$. In all cases below one basis vector from (3.2), namely that one corresponding to the vector potential, is divided, and the others multiplied by the same function $f(t, r, \varphi, z)$. The conditions of H-M conjecture are automatically fulfilled if

$$f(t, r, \varphi, z) = 1 + c_1 f_1(t, r, \varphi, z), \tag{3.5}$$

and if the function $f_1(t, r, \varphi, z)$ is a solution of the differential equation

$$G = G^{(\mu)}{}_{(\mu)} = -R = 0 \tag{3.6}$$

arising from the well known fact that for a pure electromagnetic spacetime the Einstein tensor is traceless (see e.g. [35], problem 4.16). Thus all obtained solutions have zero scalar curvature R. We shall see that the analytic form of f_1 is determined with the basis vector (3.2) collinear with the vector potential. The constant c_1 in (3.5) must naturally involve the parameter q as the limit

$$\lim_{c_1 \to 0} f(t, r, \varphi, z) = 1$$

for any regular f_1 gives original seed metric.

c) Completing the steps a) and b) we can ensure the validity of sourceless Maxwell equations. Substituting $\tilde{\boldsymbol{g}}$ into the Einstein equations we are able to fit the constant c_1 against q.

Although the simplicity of the above outlined algorithm is obviously caused by relatively high degree of symmetry characteristic for the L-C metric, this scheme might contribute to the discussion about the application of the conjecture and the character of relation between Killing vectors and electromagnetic field. No matter how surprising it may seem, this scheme is efficient even in more complicated cases discussed in chapter 4: it was successfully applied to the metric of an infinite plane (section 4.3) as well as to the γ-metric (section 4.6). The following sections of this chapter are devoted to the particular applications of the outlined scheme in case of the seed L-C metric.

3.3 The L-C solution with azimuthal magnetic field

Let us start with the Killing vector $\boldsymbol{\xi}_z$. The simplest possible choice of f in Eq. (3.5) is $f = f(r)$. Therefore, in spirit of the above discussed scheme the modified tetrad takes the form

$$\boldsymbol{\omega}^{(0)} = f(r) r^{2\sigma} \boldsymbol{dt}, \qquad \boldsymbol{\omega}^{(1)} = f(r) r^{2\sigma(2\sigma-1)} \boldsymbol{dr},$$
$$\boldsymbol{\omega}^{(2)} = f(r) C^{-1} r^{1-2\sigma} \boldsymbol{d\varphi}, \qquad \boldsymbol{\omega}^{(3)} = \frac{r^{2\sigma(2\sigma-1)}}{f(r)} \boldsymbol{dz}$$

and the corresponding line element reads as

$$\mathrm{d}s^2 = -f(r)^2 r^{4\sigma} \mathrm{d}t^2 + f(r)^2 r^{4\sigma(2\sigma-1)} \mathrm{d}r^2 + f(r)^2 C^{-2} r^{2-4\sigma} \mathrm{d}\varphi^2 + \frac{r^{4\sigma(2\sigma-1)}}{f(r)^2} \mathrm{d}z^2. \tag{3.7}$$

For the four-potential we have

$$\boldsymbol{A} = q \frac{r^{4\sigma(2\sigma-1)}}{f(r)} \boldsymbol{dz} = q r^{2\sigma(2\sigma-1)} \boldsymbol{\omega}^{(3)}.$$

The condition of traceless Einstein tensor (3.6) determines the function f

$$f(r) = 1 + c_1 r^{4\sigma(2\sigma-1)}$$

and from the Einstein equations one obtains

$$c_1 = q^2.$$

Consequently, non-zero tetrad components of Einstein tensor are

$$G_{(0)(0)} = G_{(1)(1)} = -\, G_{(2)(2)} = G_{(3)(3)} = 16\sigma^2 q^2 \frac{(2\sigma-1)^2}{r^2 f(r)^4}.$$

The electromagnetic field is of magnetic type, which can be demonstrated by the electromagnetic invariant (2.3)

$$F_{(\mu)(\nu)} F^{(\mu)(\nu)} = 32\, \frac{q^2 \sigma^2 (2\sigma-1)^2}{r^2 f(r)^2} \geq 0$$

or directly by the electromagnetic field tensor

$$F^{(1)(3)} = -B^{(2)} = \frac{4q\sigma(2\sigma-1)}{r f(r)^2}.$$

The only non-zero component of the magnetic field strength $\boldsymbol{B}$ is the azimuthal one. In accordance with the accepted physical interpretation of the L-C solution (at least for some values of σ) the line element (3.7) describes an E-M field of an infinite line source with an azimuthal magnetic field, in other words, the gravitational field of an infinite line source with an electric current.

The metric (3.7) has a one-dimensional singularity along the z-axes $r = 0$ where the Kretschmann scalar

$$
\begin{aligned}
\mathcal{R} \;=\; & \frac{64\sigma^2\left(2\sigma-1\right)^2}{f(r)^8 r^{4(4\sigma^2-2\sigma+1)}}\left[g_1(r)^4\left(4\sigma-1\right)^2\left(12\sigma^2-6\sigma+1\right)-\right.\\
& -12g_1(r)^3\sigma\left(2\sigma-1\right)\left(4\sigma-1\right)^2+\\
& +2g_1(r)^2\left(160\sigma^4-160\sigma^3+24\sigma^2+8\sigma-1\right)+\\
& \left.+12g_1(r)\sigma\left(2\sigma-1\right)+4\sigma^2-2\sigma+1\right].
\end{aligned}
$$

becomes infinite; $g_1(r) = q^2 r^{4\sigma(2\sigma-1)}$. Obviously, the only exceptions are the cases $\sigma = 0$, $\sigma = \frac{1}{2}$ for which the seed L-C metric is flat. Then also the metric (3.7) becomes flat and does not include any electromagnetic field.

Eventually, the analytic expressions for the Weyl scalars have the form

$$
\begin{aligned}
\Psi_0 = \Psi_4 \;=\; & -\frac{\left(2\sigma-1\right)\sigma}{r^2 f(r)^5 g_1(r)}\left(q^4 g_1(r)^2-1\right)\left[q^2 g_1(r)\left(24\sigma^2-10\sigma+1\right)+\right.\\
& \left.+\ \left(2\sigma+1\right)\right],\\
\Psi_2 \;=\; & \frac{\left(2\sigma-1\right)^2\sigma}{r^2 f(r)^7 g_1(r)}\left[q^{10}g_1(r)^5(4\sigma-1)+q^8 g_1(r)^4(8\sigma-3)\ -\right.\\
& \left.-2q^6 g_1(r)^3-2q^4 g_1(r)^2(4\sigma-1)-q^2 g_1(r)(4\sigma-3)+1\right].
\end{aligned}
$$

Thus the spacetime (3.7) is generally Petrov type I, special cases being:

σ	Petrov type	Weyl Scalars
$0, \frac{1}{2}$	0	all zero
$\frac{1}{4}$	D	$\Psi_0 = \Psi_4 = -3\Psi_2 = \frac{3}{16}\frac{(q^2-\sqrt{r})}{(q^2+\sqrt{r})^4}$

It should be noted that though (3.7) reminds of the general static cylindrically symmetric solution with azimuthal magnetic field [32], §20.2, Eq. 20.9a

$$
\mathrm{d}s^2 = \varrho^{2m^2}G^2\left(\mathrm{d}\varrho^2-\mathrm{d}t^2\right)+\varrho^2 G^2\mathrm{d}\varphi^2+G^{-2}\mathrm{d}z^2,
$$

where $G = C_1\varrho^m + C_2\varrho^{-m}$ and C_1, C_2, m are real constants, it does *not* belong to this class of spacetimes. This inevitably means the metric 20.9a in [32] does not represent the most general cylindrical symmetric E-M solution with azimuthal magnetic field. Putting $q = 0$, which means no electromagnetic field, one obtains the seed L-C metric.

3.4 The L-C solution with longitudinal magnetic field

Let us take the Killing vector $\boldsymbol{\xi}_\varphi$ now. This requires the tetrad

$$\begin{aligned} \boldsymbol{\omega}^{(0)} &= f(r) r^{2\sigma} \boldsymbol{dt}, & \boldsymbol{\omega}^{(1)} &= f(r) r^{2\sigma(2\sigma-1)} \boldsymbol{dr}, \\ \boldsymbol{\omega}^{(2)} &= \frac{r^{1-2\sigma}}{C f(r)} \boldsymbol{d\varphi}, & \boldsymbol{\omega}^{(3)} &= f(r) r^{2\sigma(2\sigma-1)} \boldsymbol{dz} \end{aligned} \tag{3.8}$$

and leads to the line element

$$\mathrm{d}s^2 = -f(r)^2 r^{4\sigma} \mathrm{d}t^2 + f(r)^2 r^{4\sigma(2\sigma-1)} \left[\mathrm{d}r^2 + \mathrm{d}z^2 \right] + \frac{r^{2-4\sigma}}{f(r)^2 C^2} \, \mathrm{d}\varphi^2. \tag{3.9}$$

Killing vector $\boldsymbol{\xi}_\varphi$ induces the four-potential

$$\boldsymbol{A} = \frac{q r^{2(1-2\sigma)}}{C^2 f(r)} \boldsymbol{d\varphi} = q C^{-1} r^{1-2\sigma} \boldsymbol{\omega}^{(2)}.$$

Following the scheme described in the section 3.2 one gets

$$f(r) = 1 + c_1 r^{2(1-2\sigma)}, \qquad c_1 = \frac{q^2}{C^2}. \tag{3.10}$$

and

$$G_{(0)(0)} = G_{(1)(1)} = G_{(2)(2)} = -\, G_{(3)(3)} = \frac{4q^2(2\sigma-1)^2}{C^2 f(r)^4 r^{8\sigma^2}}.$$

The electromagnetic field is again of magnetic type since

$$F_{(\mu)(\nu)} F^{(\mu)(\nu)} = \frac{8q^2 \left(2\sigma-1\right)^2}{C^2 f(r)^4 r^{8\sigma^2}} \geq 0,$$

while the only non-zero component of magnetic field strength being the longitudinal one in direction along the z-axis

$$F^{(1)(2)} = B^{(3)} = -\frac{2q(2\sigma-1)}{C f(r)^2 r^{4\sigma^2}}.$$

In accordance with the accepted physical interpretation of L-C solution (at least for some values of σ) the line element (3.9) describes an E-M field of an infinite line source with a longitudinal magnetic field, in other words, the gravitational field of an infinite line source in a Bonnor-Melvin-like universe (see e.g. [32], § 20.2, Eq. 20.10) which is responsible for the background longitudinal magnetic field. The situation reminds us of the solution described by Cataldo et al. [7] called “pencil of light in the Bonnor-Melvin Universe”. Moreover, substituting $\sigma = 0$ into (3.9) one easily comes to

$$\mathrm{d}s^2 = \left(1 + q^2 r^2 / C^2\right)^2 \left[-\mathrm{d}r^2 + \mathrm{d}r^2 + \mathrm{d}z^2 \right] + \frac{r^2}{\left(1 + q^2 r^2 / C^2\right) C^2} \, \mathrm{d}\varphi^2. \tag{3.11}$$

Evidently, this metric with $q = B_0/2$ and $C = 1$ gives the well-known B-M solution of E-M equations. Let us remind that for $\sigma = 0$, $C = 1$ the seed L-C metric reduces to Minkowski spacetime. Thus, the B-M universe can be obtained through the H-M conjecture straight from the Minkowski spacetime expressed in common cylindrical coordinates. This possibility was already mentioned by Cataldo et al. [7].

Like the seed L-C metric, the solution (3.9) has one-dimensional singularity along the z-axis, since the Kretschmann scalar

$$\begin{aligned}\mathcal{R} \;=\;& \frac{64\,(2\sigma-1)^2}{f(r)^8 r^{8\sigma(2\sigma-1)+4}}\left[g_2(r)^4\left(4\sigma^2-6\sigma+3\right)(\sigma-1)^2+\right.\\ &+6g_2(r)^3\,(2\sigma-1)\,(\sigma-1)^2-\\ &-g_2(r)^2\left(8\sigma^4-16\sigma^3-12\sigma^2+20\sigma-5\right)-\\ &\left.-\,6g_2(r)\sigma^2\,(2\sigma-1)+\sigma^2\,(4\sigma^2-2\sigma+1)\right],\end{aligned}$$

where $g_2(r) = q^2r^{2-4\sigma}/C^2$, diverges at $r = 0$. The only exception is clearly the case $\sigma = \frac{1}{2}$ for which the metric is flat and does not include any electromagnetic field.

The expressions for the Weyl scalars

$$\begin{aligned}\Psi_0=\Psi_4 \;=\;& \frac{(2\sigma-1)\,(C^2r^{4\sigma}-q^2r^2)}{C^4 f(r)^4 r^{8\sigma^2-12\sigma+2}}\left[r^{4\sigma}C^2\sigma\,(2\sigma+1)+\right.\\ &\left.+\,q^2r^2\left(2\sigma^2-5\sigma+3\right)\right],\\ \Psi_2 \;=\;& \frac{(2\sigma-1)^2\,(C^2r^{4\sigma}-q^2r^2)}{C^4 f(r)^4 r^{8\sigma^2-12\sigma+2}}\left[C^2\sigma r^{4\sigma}+q^2r^2\,(\sigma-1)\right]\end{aligned}$$

again lead to the conclusion that the metric is generally Petrov type I with the exception of algebraically special cases

σ	Petrov type	Weyl Scalars
0	D	$\Psi_0 = \Psi_4 = 3\Psi_2 = \dfrac{3q^2\,(q^2r^2 - C^2)}{C^4\,(C^2+q^2r^2)^4}$
$\frac{1}{2}$	0	all zero
1	D	$\Psi_0 = \Psi_4 = 3\Psi_2 = \dfrac{3\,(C^2r^2 - q^2)}{C^2\,(C^2r^2+q^2)^4}$

The metric (3.9) reminds of the general static cylindrically symmetric solution with longitudinal magnetic field [32], §20.2, Eq. 20.9b

$$\mathrm{d}s^2 = \varrho^{2m^2}G^2\left(\mathrm{d}\varrho^2 - \mathrm{d}t^2\right) + G^{-2}\mathrm{d}\varphi^2 + \varrho^2G^2\mathrm{d}z^2,$$

where $G = C_1\varrho^m + C_2\varrho^{-m}$ and C_1, C_2, m are real constants, but it does *not* belong to this class of spacetimes (the exception being the B-M universe (3.11)). This definitely means that the metric 20.9b in [32] does not represent the most general cylindrical symmetric E-M field with longitudinal magnetic field.

3.5 The L-C solution with radial electric field

The last from the Killing vectors (3.4) is $\boldsymbol{\xi}_t$. The tetrad

$$\begin{aligned}\boldsymbol{\omega}^{(0)} &= \frac{r^{2\sigma}}{f(r)}\boldsymbol{dt}, & \boldsymbol{\omega}^{(1)} &= f(r)r^{2\sigma(2\sigma-1)}\boldsymbol{dr},\\ \boldsymbol{\omega}^{(2)} &= f(r)C^{-1}r^{1-2\sigma}\boldsymbol{d\varphi}, & \boldsymbol{\omega}^{(3)} &= f(r)r^{2\sigma(2\sigma-1)}\boldsymbol{dz}\end{aligned}$$

determines the metric

$$\mathrm{d}s^2 = -\frac{r^{4\sigma}}{f(r)^2}\mathrm{d}t^2 + f(r)^2 r^{4\sigma(2\sigma-1)}\left[\mathrm{d}r^2 + \mathrm{d}z^2\right] + f(r)^2 C^{-2} r^{2-4\sigma}\mathrm{d}\varphi^2, \tag{3.12}$$

where

$$f(r) = 1 + c_1 r^{4\sigma},$$

and

$$c_1 = -q^2.$$

The non-zero tetrad components of Einstein tensor then read

$$G_{(0)(0)} = -G_{(1)(1)} = G_{(2)(2)} = G_{(3)(3)} = \frac{16q^2\sigma^2 r^{-8\sigma^2+8\sigma-2}}{f(r)^4}.$$

The vector potential

$$\boldsymbol{A} = -\frac{q r^{4\sigma}}{f(r)}\boldsymbol{dt} = -q r^{2\sigma}\boldsymbol{\omega}^{(0)}$$

sets the field of electric type

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = -32\,\frac{q^2\sigma^2 r^{-8\sigma^2+8\sigma-2}}{f(r)^4} \leq 0$$

with non-zero radial component of electric field strength

$$F^{(0)(1)} = E^{(1)} = -\frac{4q\sigma r^{-4\sigma^2+4\sigma-1}}{f(r)^2}.$$

One can conclude that the metric (3.12) describes E-M field of a charged infinite line source.

Unlike the seed metric (3.1) and the solutions (3.7), (3.9), the spacetime (3.12) contains not only one dimensional singularity along the z-axes but also a singularity at a radial distance r_s for which

$$f(r_s) = 1 - q^2 r_s^{4\sigma} = 0.$$

The Kretschmann scalar

$$\begin{aligned}\mathcal{R} &= \frac{64\sigma^2}{f(r)^8 r^{16\sigma^2-8\sigma+4}}\Big[g_3(r)^4\left(4\sigma^2+2\sigma+1\right)(2\sigma+1)^2 + \\ &\quad + 12 g_3(r)^3\sigma\,(2\sigma+1)^2 - 2g_3(r)^2\left(16\sigma^4 - 48\sigma^2 + 1\right) - \\ &\quad - 12 g_3(r)\sigma\,(2\sigma+1)^2 + (2\sigma-1)^2\left(4\sigma^2 - 2\sigma + 1\right)\Big],\end{aligned}$$

where $g_3(r) = q^2 r^{4\sigma}$, proves that this surface represents physical singularity that cannot be removed by any coordinate transformation. The interpretation of this singularity represents an interesting, open problem.

The solution (3.12) is generally Petrov type I with Weyl scalars

$$\begin{aligned}\Psi_0 = \Psi_4 &= (2\sigma-1)(2\sigma+1)\,\sigma r^{-8\sigma^2+4\sigma-2}\frac{1+q^2 r^{4\sigma}}{f(r)^3}, \\ \Psi_2 &= -\frac{\sigma r^{-8\sigma^2+4\sigma-2}}{f(r)^4}\left[q^2 r^{4\sigma}(2\sigma+1)^2 - (2\sigma-1)^2 - 8q^2\sigma r^{4\sigma}\right];\end{aligned}$$

the only exceptions belonging to other Petrov classes are

σ	Petrov type	Weyl Scalars
0	0	all zero
$\dfrac{1}{2}$	D	$\Psi_2 = -\,2\,\dfrac{q^2\,(1+q^2r^2)}{(1-q^2r^2)^4}$
$-\,\dfrac{1}{2}$	D	$\Psi_2 = -\,2\,\dfrac{(q^2+r^2)}{(q^2-r^2)^4}$

The metric (3.12) represents a special case of the general class of cylindrically symmetric solutions with radial electric field given in [32], §20.2, Eq. 20.9c. The line-element of this general class reads as

$$\mathrm{d}s^2 = \varrho^{2m^2} G^2 \left(\mathrm{d}\varrho^2 + \mathrm{d}z^2\right) + \varrho^2 G^2 \mathrm{d}\varphi^2 - G^{-2}\mathrm{d}t^2,$$

where $G = C_1\varrho^m + C_2\varrho^{-m}$ and C_1, C_2, m are real constants. Nevertheless, the approach within the framework of the H-M conjecture provides a promising possibility for its physical interpretation and for understanding the nature of sources (at least for some values of σ).

3.6 The magnetovacuum solution for $\sigma = -1/2$

Before proceeding to the Killing vector $\boldsymbol{\xi}_{-1/2}$ several explanatory notes should be added. The vector $\boldsymbol{\xi}_{-1/2}$ obviously generates rotation in the $\varphi - z$ plane, so it is geometrically equivalent to $\boldsymbol{\xi}_\varphi$ and should lead to magnetic E-M field. The cylindrical coordinates used so far are very convenient for the expression of $\boldsymbol{\xi}_\varphi$. To follow the scheme outlined in section 3.2 one should preferably start with the coordinate transformation

$$\varphi = CX\cos Y, \qquad z = X\sin Y,$$

which turn the L-C metric (3.1) for $\sigma = -\frac{1}{2}$ into the form

$$\mathrm{d}s^2 = -\,\frac{\mathrm{d}t^2}{r^2} + r^4\mathrm{d}r^2 + r^4\mathrm{d}X^2 + r^4X^2\mathrm{d}Y^2$$

and the Killing vector

$$\boldsymbol{\xi}_{-1/2} = \partial_Y = r^4X^2\boldsymbol{dY}.$$

Now in analogy with all preceding cases let us take the basis

$$\boldsymbol{\omega}^{(0)} = \frac{F(X,r)}{r}\boldsymbol{dt}, \qquad \boldsymbol{\omega}^{(1)} = F(X,r)r^2\boldsymbol{dr},$$
$$\boldsymbol{\omega}^{(2)} = F(X,r)r^2\boldsymbol{dX}, \qquad \boldsymbol{\omega}^{(3)} = \frac{r^2X}{F(X,r)}\boldsymbol{dY}$$

inducing a metric

$$\mathrm{d}s^2 = -\,\frac{F(X,r)^2}{r^2}\mathrm{d}t^2 + F(X,r)^2r^4\mathrm{d}r^2 + F(X,r)^2r^4\mathrm{d}X^2 + \frac{r^4X^2}{F(X,r)^2}\mathrm{d}Y^2$$

and set the four-potential

$$\boldsymbol{A} = q\frac{r^4X^2}{F(X,r)}\boldsymbol{dY}.$$

The solution of Einstein and sourceless Maxwell equations then provides

$$F(X,r) = 1 + q^2X^2r^4.$$

Although the coordinates (t, r, X, Y) are extremely suitable for calculation of tensor components and solving E-M equations, the cylindrical coordinates (t, r, φ, z) fit better the aim of physical interpretation, namely because of the evident relation to the L-C seed metric. Therefore, let us consequently transform all above computed objects back into the cylindrical coordinates. We obtain

$$\boldsymbol{A} = q\frac{r^4}{Cf(r,\varphi,z)}\left(-z\,\boldsymbol{d\varphi} + \varphi\,\boldsymbol{dz}\right) = qr^2\sqrt{\varphi^2/C^2+z^2}\,\boldsymbol{\omega}^{(3)},$$

$$\begin{aligned}
\boldsymbol{\omega}^{(0)} &= \frac{f(r,\varphi,z)}{r}\,\boldsymbol{dt}, \quad \boldsymbol{\omega}^{(1)} = f(r,\varphi,z)r^2\,\boldsymbol{dr},\\
\boldsymbol{\omega}^{(2)} &= \frac{r^2f(r,\varphi,z)}{\sqrt{\varphi^2/C^2+z^2}}\left(\frac{\varphi}{C^2}\,\boldsymbol{d\varphi} + z\,\boldsymbol{dz}\right),\\
\boldsymbol{\omega}^{(3)} &= \frac{r^2}{\sqrt{\varphi^2/C^2+z^2}f(r,\varphi,z)}\left(-\frac{z}{C}\,\boldsymbol{d\varphi} + \frac{\varphi}{C}\,\boldsymbol{dz}\right);
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}s^2 \;=\; & -\frac{f(r,\varphi,z)^2}{r^2}\mathrm{d}t^2 + f(r,\varphi,z)^2r^4\mathrm{d}r^2 +\\
& + \frac{r^4}{\varphi^2/C^2+z^2}\left[f(r,\varphi,z)^2\left(\frac{\varphi}{C^2}\,\mathrm{d}\varphi + z\,\mathrm{d}z\right)^2 + \right.\\
& \left. + \frac{1}{f(r,\varphi,z)^2}\left(-\frac{z}{C}\,\mathrm{d}\varphi + \frac{\varphi}{C}\,\mathrm{d}z\right)^2\right],
\end{aligned} \tag{3.13}$$

where

$$f(r,\varphi,z) = 1 + q^2\left(\varphi^2/C^2+z^2\right)r^4.$$

In a standard way one finds non-zero tetrad components of the Einstein tensor

$$\begin{aligned}
G_{(0)(0)} = G_{(3)(3)} &= \frac{4q^2\left[4(\varphi^2/C^2+z^2)+r^2\right]}{r^2f(r,\varphi,z)^4},\\
G_{(1)(1)} = -G_{(2)(2)} &= \frac{4q^2\left[4(\varphi^2/C^2+z^2)-r^2\right]}{r^2f(r,\varphi,z)^4},\\
G_{(1)(2)} = G_{(2)(1)} &= \frac{16q^2\sqrt{\varphi^2/C^2+z^2}}{rf(r,\varphi,z)^4}.
\end{aligned}$$

The electromagnetic field is of magnetic type, since

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = \frac{8q^2\left[4\left(\varphi^2/C^2+z^2\right)+r^2\right]}{r^2f(r,\varphi,z)^4} \geq 0,$$

the tetrad components of the magnetic field strength read as

$$F^{(1)(3)} = -B^{(2)} = \frac{4q\sqrt{\varphi^2/C^2+z^2}}{rf(r,\varphi,z)^2}, \quad F^{(2)(3)} = B^{(1)} = \frac{2q}{f(r,\varphi,z)^2}.$$

The physical interpretation of (3.13) is rather ambiguous because of the negative value of σ. As we have already mentioned in section 3.1, in case of negative σ the problem of physical cylindrical symmetric sources was not solved in a satisfactory way even for the seed L-C metric. Bonnor (see references in [6]) has proved that the seed L-C metric for $\sigma = -1/2,\ C = 1$ is locally isometric to Taub's plane solution. Therefore, a more suitable alternative seems to be a point of view preferred by Wang at all [55], that both the L-C solution in case of negative σ and the metric (3.13) are plane symmetric.

The Kretschmann scalar

$$\begin{aligned}\mathcal{R} \;=\; & \frac{64}{f(r,\varphi,z)^8 r^{12}}\Big[3q^8r^{16}\left(\varphi^2/C^2+z^2\right)^2\times\\ & \times\left(21\varphi^4/C^4+42\varphi^2z^2/C^2+6\varphi^2r^2/C^2+6r^2z^2+21z^4+r^4\right)-\\ & -6q^6r^{12}\left(\varphi^2/C^2+z^2\right)\left(18\varphi^4/C^4+7\varphi^2r^2/C^2+36\varphi^2z^2/C^2+\right.\\ & \left.+7r^2z^2+18z^4+r^4\right)+q^4r^8\left(62z^4+124\varphi^2z^2/C^2+62\varphi^4/C^4+\right.\\ & \left.+46\varphi^2r^2/C^2+46r^2z^2+5r^4\right)+6q^2r^4\left(2z^2-r^2+2\varphi^2/C^2\right)+3\Big].\end{aligned}$$

diverges at $r = 0$; this one-dimensional singularity along the z-axis might indicate the location of the infinite line source. The Petrov type is I with Weyl scalars

$$\begin{aligned}\Psi_0=\Psi_4 &= -12\,\frac{q^2\left(\varphi^2/C^2+z^2\right)\left[q^2r^4\left(\varphi^2/C^2+z^2\right)-1\right]}{r^2f(r,\varphi,z)^4}\\ \Psi_1=-\Psi_3 &= -6\,\frac{q^2\sqrt{\varphi^2/C^2+z^2}\left[q^2r^4\left(\varphi^2/C^2+z^2\right)-1\right]}{rf(r,\varphi,z)^4},\\ \Psi_2 &= \frac{2}{r^6f(r,\varphi,z)^4}\Big\{\left[q^2r^4\left(\varphi^2/C^2+z^2\right)-1\right]\times\\ &\quad\left[3q^2r^4\left(\varphi^2/C^2+z^2\right)-q^2r^6+1\right]\Big\}.\end{aligned}$$

3.7 The electrovacuum solution for $\sigma = 1/4$

$\boldsymbol{\xi}_{1/4}$, the last Killing vector listed in section 3.1, characterizes a boost in the $t-\varphi$ plane. As in the previous section, one should preferably perform coordinate transformation

$$t = X\cosh Y, \qquad \varphi = CX\sinh Y$$

to simplify the calculus. This transformation does not map the whole spacetime but only the interior of the light cone $t^2-\varphi^2 > 0$. The rest of the spacetime can be mapped analogously when we interchange the hyperbolic functions and then one should formulate suitable boundary conditions to couple those maps smoothly together. Here we shall restrict ourselves to the interior of the light cone only. After substituting $\sigma = 1/4$ Eq. (3.1) turns into

$$\mathrm{d}s^2 = -r\mathrm{d}X^2+\frac{\mathrm{d}r^2}{\sqrt{r}}+rX^2\mathrm{d}Y^2+\frac{\mathrm{d}z^2}{\sqrt{r}}.$$

It is worth mentioning that for $\sigma = 1/4$ the L-C metric represents a transformation of one of the Kinnersley's type D metric [28] (his Case IVB with his $C = 1$). For the Killing vector one gets $\boldsymbol{\xi}_{1/4} = \partial_Y = rX^2\boldsymbol{dY}$. The choice of the tetrad

$$\boldsymbol{\omega}^{(0)} = \sqrt{r}F(X,r)\boldsymbol{dX}, \qquad \boldsymbol{\omega}^{(1)} = \frac{F(X,r)}{r^{1/4}}\boldsymbol{dr},$$
$$\boldsymbol{\omega}^{(2)} = \frac{\sqrt{r}X}{F(X,r)}\boldsymbol{dY}, \qquad \boldsymbol{\omega}^{(3)} = \frac{F(X,r)}{r^{1/4}}\boldsymbol{dz}$$

gives the metric

$$\mathrm{d}s^2 = -F(X,r)^2 r\mathrm{d}X^2 + \frac{F(X,r)^2}{\sqrt{r}}\mathrm{d}r^2 + \frac{rX^2}{F(X,r)^2}\mathrm{d}Y^2 + \frac{F(X,r)^2}{\sqrt{r}}\mathrm{d}z^2,$$
$$F(X,r) = 1 + q^2X^2r;$$

and set the four-potential

$$\boldsymbol{A} = q\frac{rX^2}{F(X,r)}\boldsymbol{dY}.$$

Returning back to the cylindrical coordinates one obtains the vector potential

$$\boldsymbol{A} = q\frac{r}{Cf(t,r,\varphi)}\left(-\varphi\,\boldsymbol{dt} + t\boldsymbol{d\varphi}\right) = q\sqrt{r}\sqrt{t^2 - \varphi^2/C^2}\,\boldsymbol{\omega}^{(2)},$$

the basis tetrad

$$\begin{aligned}
\boldsymbol{\omega}^{(0)} &= \frac{\sqrt{r}f(t,r,\varphi)}{\sqrt{t^2-\varphi^2/C^2}}\left(t\,\boldsymbol{dt} - \frac{\varphi}{C^2}\,\boldsymbol{d\varphi}\right), & \boldsymbol{\omega}^{(1)} &= \frac{f(t,r,\varphi)}{r^{1/4}}\,\boldsymbol{dr},\\
\boldsymbol{\omega}^{(2)} &= \frac{\sqrt{r}}{\sqrt{t^2-\varphi^2/C^2}f(t,r,\varphi)}\left(-\frac{\varphi}{C}\,\boldsymbol{dt} + \frac{t}{C}\,\boldsymbol{d\varphi}\right), & \boldsymbol{\omega}^{(3)} &= \frac{f(t,r,\varphi)}{r^{1/4}}\,\boldsymbol{dz},
\end{aligned} \tag{3.14}$$

and the line element

$$\begin{aligned}
\mathrm{d}s^2 \;=\; & \frac{r}{t^2-\varphi^2/C^2}\left[-f(t,r,\varphi)^2\left(t\mathrm{d}t - \frac{\varphi}{C^2}\mathrm{d}\varphi\right)^2 + \right.\\
& \left. + \frac{1}{f(t,r,\varphi)^2}\left(-\frac{\varphi}{C}\mathrm{d}t + \frac{t}{C}\mathrm{d}\varphi\right)^2\right] + \frac{f(t,r,\varphi)^2}{\sqrt{r}}\left(\mathrm{d}r^2 + \mathrm{d}z^2\right),
\end{aligned} \tag{3.15}$$

where.

$$f(t,r,\varphi) = 1 + q^2\left(t^2 - \varphi^2/C^2\right)r.$$

Non-zero tetrad components of the Einstein tensor are

$$\begin{aligned}
G_{(0)(0)} = G_{(1)(1)} &= \frac{q^2\left(4\sqrt{r} + t^2 - \varphi^2\right)}{\sqrt{r}f(t,r,\varphi)^4},\\
G_{(2)(2)} = -G_{(3)(3)} &= \frac{q^2\left(-4\sqrt{r} + t^2 - \varphi^2\right)}{\sqrt{r}f(t,r,\varphi)^4},\\
G_{(0)(1)} = G_{(1)(0)} &= -\frac{4q^2\sqrt{t^2-\varphi^2}}{r^{1/4}f(t,r,\varphi)^4}.
\end{aligned}$$

This time the electromagnetic field is neither purely electric, nor purely magnetic but its type is different at various places and at various time, which can be demonstrated by the invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = -\frac{2q^2\left[4\sqrt{r} - \left(t^2 - \varphi^2/C^2\right)\right]}{\sqrt{r}f(t,r,\varphi)^4}$$

and components of the electromagnetic field tensor

$$F^{(0)(2)} = E^{(2)} = \frac{2q}{f(t,r,\varphi)^2}, \quad F^{(1)(2)} = B^{(3)} = \frac{q\sqrt{t^2-\varphi^2}}{r^{1/4} f(t,r,\varphi)^2}.$$

This rather strange behaviour originates in the fact that the tetrad (3.14) is carried by an observer moving round the infinite line source in azimuthal direction, that means, rotating round z-axis. In this sense the indefinite character of electromagnetic field can be understood as a special-relativistic effect. The E-M field (3.15) can be interpreted as an infinite line source with linear density $2.5{\cdot}10^{27}\,\mathrm{g{\cdot}cm^{-1}}$ in the external electromagnetic field, part of which is analogous to the B-M longitudinal magnetic background. The solution (3.15) is non-stationary and in contrast to the seed L-C metric it is neither cylindrically, nor axially symmetric.

In the interior of the light cone $t^2 - \varphi^2/C^2 > 0$ the metric (3.15) has again one-dimensional singularity at $r = 0$ where the Kretschmann scalar

$$\begin{aligned}\mathcal{R} = {} & \frac{1}{4f(t,r,\varphi)^8 r^{7/2}} \Big[3q^8 \left(t^2 - \varphi^2/C^2\right) \Big(21\varphi^4 r^{9/2}/C^4 - \\ & - 42\varphi^2 t^2 r^{9/2}/C^2 + 96\varphi^2 r^5/C^2 + 256 r^{11/2} + 21 t^4 r^{9/2} - 96 t^2 r^5\Big) - \\ & - 12 q^6 \left(t^2 - \varphi^2/C^2\right) \Big(9\varphi^4 r^{7/2}/C^4 + 56\varphi^2 r^4/C^2 - 18\varphi^2 t^2 r^{7/2}/C^2 + \\ & + 9t^4 r^{7/2} - 56 r^4 t^2 + 128 r^{9/2}\Big) + 2q^4 \Big(368\varphi^2 r^3/C^2 - 368 r^3 t^2 + \\ & + 31\varphi^4 r^{5/2}/C^4 + 31 r^{5/2} t^4 + 640 r^{7/2} - 62\varphi^2 r^{5/2} t^2/C^2\Big) + \\ & + 12 q^2 \left(t^2 r^{3/2} - \varphi^2 r^{3/2}/C^2 + 8r^2\right) + 3\sqrt{r}\,\Big].\end{aligned}$$

becomes infinite. The Petrov type is I with the Weyl scalars

$$\begin{aligned}\Psi_0 = \Psi_4 &= \frac{3}{4}\frac{q^2\left(t^2-\varphi^2/C^2\right)\left[q^2 r\left(t^2-\varphi^2/C^2\right)-1\right]}{\sqrt{r} f(t,r,\varphi)^4}, \\ \Psi_1 = -\Psi_3 &= -\frac{3q^2}{2}\frac{q^2 r\left(t^2-\varphi^2/C^2\right)^2 - \left(t^2-\varphi^2/C^2\right)}{r^{1/4}\sqrt{t^2-\varphi^2/C^2}\, f(t,r,\varphi)^4}, \\ \Psi_2 &= \frac{1}{8r^{3/2} f(t,r,\varphi)^4}\Big[3q^4 r^2\left(t^2-\varphi^2/C^2\right)^2 + 16 q^4 r^{5/2}\left(t^2-\varphi^2/C^2\right) - \\ & \quad - 16 q^2 r^{3/2} - 2q^2 r\left(t^2-\varphi^2/C^2\right) - 1\Big].\end{aligned}$$

3.8 The L-C solution with longitudinal electric field

In preceding sections all Killing vectors of the L-C metric were exhausted to generate E-M field. As a by-product of those calculations one more E-M field was found, or surprisingly, new possible interpretation was assigned to the metric (3.9) derived in section 3.4. It should be pointed out that though the next steps and considerations follow the scheme of the H-M conjecture described above, there is a key difference: we do not employ any Killing vector of the seed L-C metric as a vector four-potential. On the other hand, one should emphasize that this fact does not contradict the conjecture, which has never been considered as the only possibility of generating E-M fields.

Let us take the boost vector potential

$$\boldsymbol{A} = q\left(z\boldsymbol{dt} - t\boldsymbol{dz}\right) = \frac{q}{f(r)}\left(z r^{-2\sigma}\boldsymbol{\omega}^{(0)} - t r^{2\sigma}\boldsymbol{\omega}^{(3)}\right)$$

with tetrad (3.8) inducing metric (3.9) and with the function $f(r)$ in the form (3.10). Solving E-M equations one derives the constant c_1

$$c_1 = \frac{q^2}{(2\sigma-1)^2}$$

and components of the Einstein tensor

$$G_{(0)(0)} = G_{(1)(1)} = G_{(2)(2)} = -\,G_{(3)(3)} = \frac{4q^2}{r^{8\sigma^2} f(r)^4}.$$

The electromagnetic invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = -\,\frac{8q^2C^2\left(2\sigma-1\right)^2}{r^{8\sigma^2} f(r)^4} < 0,$$

so the electromagnetic field represents an electric type with longitudinal electric field strength oriented along the z-axis

$$F^{(0)(3)} = E^{(3)} = -\,\frac{2q}{f(r)^2 r^{4\sigma^2}}.$$

Thus we have generated E-M field of an infinite line source in a universe filled with the longitudinal electric field. The source of the electric field can be hardly identified with the charge distribution along the z-axis. Therefore the electric field should be understood like a background which has the same role as the magnetic field in the B-M universe. Moreover, substituting $\sigma = 0$, $C = 1$ one is left with the metric (3.11), the B-M universe. The conclusion is straightforward: the B-M universe need not necessary represent a magnetic E-M field but also an electric one or their combination.

The Kretschmann scalar

$$\begin{aligned}
\mathcal{R} \;=\; & \frac{64\left(2\sigma-1\right)^2}{f(r)^8 r^{8\sigma(2\sigma-1)+4}}\Big[g_4(r)^4\left(\sigma-1\right)^2\left(4\sigma^2-6\sigma+3\right)+ \\
& + 6g_4(r)^3\left(2\sigma-1\right)\left(\sigma-1\right)^2 - \\
& - g_4(r)^2\left(8\sigma^4-16\sigma^3-12\sigma^2+20\sigma-5\right) - \\
& - 6g_4(r)\sigma\left(2\sigma-1\right) + \sigma^2\left(4\sigma^2-2\sigma+1\right)\Big],
\end{aligned}$$

where $g_4(r) = q^2 r^{2-4\sigma}/\left(2\sigma-1\right)^2$ is again singular at $r = 0$ with the exception $\sigma = 1/2$, in which case the spacetime is flat and does not include any electric field.

Analogously to (3.9) the metric is Petrov type I with Weyl scalars

$$\begin{aligned}
\Psi_0 = \Psi_4 \;=\; & -\frac{q^2r^2 - r^{4\sigma}\left(2\sigma-1\right)^2}{r^{8\sigma^2+16\sigma-10}(2\sigma-1)^3 f(r)^4}\Big[r^{4\sigma}\left(8\sigma^4-4\sigma^3-2\sigma^2+\sigma\right)+ \\
& + q^2r^2\left(2\sigma^2-5\sigma+3\right)\Big], \\
\Psi_2 \;=\; & -\frac{q^2r^2 - r^{4\sigma}\left(2\sigma-1\right)^2}{r^{8\sigma^2+16\sigma-10}(2\sigma-1)^2 f(r)^4}\Big[r^{4\sigma}\sigma\left(2\sigma-1\right)^2 + q^2r^2\left(\sigma-1\right)\Big],
\end{aligned}$$

algebraic special cases are summarized in the following table:

σ	Petrov type	Weyl Scalars
0	D	$\Psi_0 = \Psi_4 = 3\Psi_2 = \dfrac{3q^2\,(qr+1)\,(qr-1)}{(q^2r^2+1)^4}$
$\dfrac{1}{2}$	0	all zero
1	D	$\Psi_0 = \Psi_4 = 3\Psi_2 = \dfrac{3\,(r^2-q^2)}{(r^2+q^2)^4}$

3.9 Superposition of E-M fields

The fact that the metric 3.9 can be interpreted as a solution both with a longitudinal magnetic field (section 3.4) or with a longitudinal electric field (section 3.4) leads straightforwardly to another problem: the superposition of various E-M fields. For some special cylindrically symmetric E-M fields it was successfully done by Safko [49], whose solution can be interpreted as a superposition of the azimuthal and longitudinal E-M fields (the line element of which is identical neither with (3.7) or with (3.9)). This question becomes even more interesting in connection with the H-M conjecture. Recalling the well-known fact that a linear combination of Killing vectors gives a Killing vector again, one would expect that applying H-M conjecture to the linear combination of Killing vectors he should come to a new Einstein-Maxwell field.

This assumption seems to be very reasonable especially in the original formulation of the conjecture proposed by Horský and Mitskievitch [22], according to which a vector four-potential $\boldsymbol{A}$ differs from the Killing vector of the seed metric only by a multiplicative constant. Then there would be no logical objection why a linear combination of Killing vectors, i.e. another Killing vector, should not lead to a some E-M field.

The Cataldo's et al. formulation quoted in section 2.1 makes the connection between the four-potentials of "charged" metrics and the Killing vectors of the seed metrics rather loose. This better reflects our experience, that every linear combination of Killing vectors need not generate an E-M field; at least in the sense, that such E-M fields have not been found because of the complexity of the obtained E-M equations. Let us take solutions (3.7) and (3.12) as an example. They correspond to an intuitive classical analogy of an infinite line-mass either with an electric current or with some charge distribution along the z-axis. It would be very convenient to have a general superposition of those two E-M fields, but we have none so far. This of course does not prove that such E-M does not exist at all.

Experience based on the study of solutions listed in section 2.2 also supports the conclusion, that only some Killing vectors from the whole Killing vector space are preferred as possible generators of the E-M fields. On the other hand, the number of solutions discussed in connection with the H-M conjecture is too small to make any definitive conclusion.

This question arises also in connection with the inverse problem (see section 2.3). Štefaník and Horský [51, 52] derived, that four-potentials of the "charged" Chitre et al. and Van den Berg-Wils solutions each coincide with a unique linear combination of the Killing vectors of the corresponding seed metrics. Then we may ask: why namely this linear combination leads to an E-M field?

Leaving this crucial and conceptual questions unanswered we can conclude that under special conditions the superposition is not only possible but at the same time nearly trivial. As a first example we can take the conformally flat E-M field (2.6) for which the electric and magnetic fields are collinear. Similarly, the longitudinal magnetic field studied in section 3.4 is collinear with the longitudinal electric field obtained in section 3.8. Moreover, the expressions for the line-elements in these cases differ only in the form of the constant c_1 figuring in the function $f(r)$. Substituting

$$c_1 = \frac{q_1^2}{C^2} + \frac{q_2^2}{(2\sigma - 1)^2}$$

into 3.10 and taking the four-potential

$$\boldsymbol{A} = \frac{q_1 r^{2(1-2\sigma)}}{C^2 f(r)}\, \boldsymbol{d\varphi} + q_2\, (z\boldsymbol{dt} - t\boldsymbol{dz})$$

we obtain a superposed E-M field

$$\mathrm{d}s^2 = -f(r)^2 r^{4\sigma}\mathrm{d}t^2 + f(r)^2 r^{4\sigma(2\sigma-1)}\left[\mathrm{d}r^2 + \mathrm{d}z^2\right] + \frac{r^{2-4\sigma}}{f(r)^2 C^2}\,\mathrm{d}\varphi^2, \tag{3.16}$$

where

$$f(r) = 1 + c_1 r^{2(1-2\sigma)}.$$

The solution (3.16) now includes both longitudinal electric field and longitudinal magnetic one and may be taken as a superposition of those two fields. The sign of the electromagnetic invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = -\frac{8\,(2\sigma-1)^2}{r^{8\sigma^2} f(r)^4}\left[\frac{q_2^2}{(2\sigma-1)^2} - \frac{q_1^2}{C^2}\right]$$

is indefinite and sets the conditions under which the field is of electric or magnetic type. The components of the Einstein tensor, the Riemann tensor as well as expressions for the Kretschmann and the Weyl scalars can be easily obtained from the corresponding ones in section 3.4 performing the substitution

$$\frac{q^2}{C^2} \longrightarrow \frac{q_1^2}{C^2} + \frac{q_2^2}{(2\sigma-1)^2}.$$

3.10 Radial geodesic motion

Geodesics provides various useful information about the physical properties of the studied spacetimes and about the character of the source. In this chapter the radial geodesics were found only for the static cylindrically symmetric cases, namely the metrics (3.1), (3.7), (3.9) and (3.12) treated in sections 3.1, 3.3, 3.4 and 3.5 respectively and also for the solution derived in the section 3.8. In all these cases the metric coefficients depend only on the radial coordinate r, so one has three integrals of motion at his disposal: the covariant coordinate components of particles four-velocity u_t, u_φ, u_z defined in the common way (here we use the fact that all the metrics are diagonal)

$$u_t = g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau}, \quad u_\varphi = g_{\varphi\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}, \quad u_z = g_{zz}\frac{\mathrm{d}z}{\mathrm{d}\tau},$$

where τ is the particle's proper time. The normalization condition

$$u_\mu u^\mu = g^{tt}u_t^2 + g_{rr}(u^r)^2 + g^{\varphi\varphi}u_\varphi^2 + g^{zz}u_z^2 = -1$$

enables us to express the square of the contravariant radial four-velocity component

$$(u^r)^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \frac{1}{g_{rr}}\left(-1 - g^{tt}u_t^2 - g^{\varphi\varphi}u_\varphi^2 - g^{zz}u_z^2\right).$$

The purely radial motion (there is, evidently, no dragging effect) is set by putting $u_\varphi = u_z = 0$. Here, unfortunately, it is not possible to introduce an effective potential independent of the particle energy per unit mass $-u_0$. Therefore, we use the absolute value of radial velocity instead. The scale of the radial velocity is not important for qualitative discussion, and thus it is not explicitly introduced in the drawings. The boundary between the region with zero and non-zero radial velocity physically determines the turning points for radial motion. To detect the position of the turning points more exactly, the contour lines are drawn in the base plane of each figure. The top flat part of the plots corresponds to regions with high radial velocities.

Each figure is related to a different spacetime and includes four subplots. These subplots correspond with two different values of particle energy (first and second row of subplots), and with weaker or stronger electromagnetic field (left and right column respectively). In this way we can illustrate the influence of the electromagnetic field on radial geodesics and compare the situation with motion in the seed L-C metric. The first value of energy ($u_0 = 1$) characterizes a particle at rest in Minkowski spacetime.

The plots in Fig. 3.2 represent the dependence of the $|u^r|$ on r for various values of σ, that means, for different spacetimes from the class of solutions (3.7). When the spacetime is flat ($\sigma = 0$) then the radial motion with constant energy must result in constant radial velocity which equals zero in cases (a), (b) and is non-zero in cases (c),(d) (the top of the "ridge"). While for negative σ (which is probably not relevant to any real situation) the singularity is not attractive and particle is kept at some distance from the z-axis, for $\sigma > 0$ the radial velocity rapidly increases towards the singularity with an evident attractive effect. The Figs. 3.2(a) and (c) illustrating the situation in presence of weak magnetic field are qualitatively identical to appropriate plots for the seed L-C solution (3.1).

Quite an analogous situation can be found in Fig. 3.3 belonging to the solution (3.9) with longitudinal magnetic field which degenerates to flat spacetime for $\sigma = 1/2$. Apparently, one should expect that in this case we again recognize the motion with a constant radial velocity in the plots. There is, however, an important difference originating in the form of the seed L-C metric. Evidently, the metric (3.1) for $\sigma = 1/2$ turns into

$$\mathrm{d}s^2 = -r^2\mathrm{d}t^2 + \mathrm{d}r^2 + \mathrm{d}\varphi^2 + \mathrm{d}z^2$$

with interchanged components g_{tt} and $g_{\varphi\varphi}$ compared to the Minkowski spacetime; it rather corresponds to the frame of an accelerated observer. Such an observer certainly will not measure a constant radial velocity for a considered radial motion. On the other hand, comparing Figs. 3.2 and 3.3, subplots (a), (c), we can see there is no crucial difference between the motion in a weak azimuthal and longitudinal magnetic field, the positions of turning points nearly coincide. Some differences can be detected in stronger fields (Figs. 3.2 and 3.3, subplots (b), (d)).

The plots become slightly more complicated in presence of electric field in Fig. 3.4. The presence of another singularity in (3.12) results in the fact that the radial geodesic motion of particles with given energy is restricted to two separated regions perceptible in all subplots (a)-(d). In case of negative σ the singularity gets an attractive character (the parts with increasing radial velocity at the back of subplots). Moreover, in case of positive σ and stronger electric field (subplots (b),(d)) the radial motion is possible also at larger radial distances from the z-axis (compare to corresponding subplots in Figs. 3.2,3.3 which are different). In subplots 3.4(a),(c) we can again recognize motion with a constant radial velocity for $\sigma = 0$, when (3.12) becomes flat.

The solution including the longitudinal electric field (section 3.8) has many features in common with the corresponding magnetic one (3.9). Comparing Figs. 3.3 and 3.5 one finds out that the electric case in Fig. 3.5 differs in the particular detail that for the case of flat spacetime $\sigma = 1/2$ the radial velocity equals zero and the plane of constant $\sigma = 1/2$ strictly divides the surfaces on all subplots into two parts. The subplots 3.5(a), (c) for a weak longitudinal electric field are analogous to those ones corresponding to a weak longitudinal magnetic field in Fig. 3.3(a), (c).

There should be stressed one more interesting point in connection with Fig. 3.5(b). For stronger electromagnetic field, even for positive σ, the singularity qualitatively changes its behaviour: there is a turning point close to the z-axis so that the particle cannot reach the singularity (see the right part of the subplot (b)). This might be another argument supporting the problematic interpretation of the L-C solution for $\sigma > 1$ (and consequently, all the solutions generated from the L-C metric in this paper).

All the above mentioned general characteristics of the geodesics motion are confirmed by four particular examples in Fig. 3.6(a)-(d). Plots (a) and (b) show bound trajectories typical for negative values of σ; the trajectories differ only in the energy integral of motion u_0. The value $q = 0.1$ is chosen the same as in the plots (a),(c) in Figs. 3.2–3.5, where one can trace a limited region with non-zero radial velocity. The motion reminds the the situation in the cylindrical potential well with a repulsive barrier protecting the z-axis (compare with the Newtonian analogy in Fig. 3.7(a)). Although the plots are drawn for the L-C solution with the longitudinal magnetic field, comparing Figs. 3.2–3.5(a), (c) one concludes, that similar trajectories could be found in case of other cylindrically symmetric solutions discussed in this section.

The trajectory in Fig. 3.6(c) is apparently not different from the previous ones, but it corresponds to the positive value $\sigma = 1$ and its shape is in qualitative agreement with the rightmost part of the Fig. 3.3(d): the singularity has again an repulsive character and does not allow the particle to reach the z-axis at $r = 0$. The trajectory is nearly closed and one can found several fixed points through which the particle has passes in each turn.

The plot in Fig. 3.6(d) then characterizes the repulsive singularity that arises in the solution with radial electric field as it can be seen by an exterior observer. The particle thrown into the z axis is repelled back to regions located far from the z-axis (confront with Fig. 3.4(d)). One should also mind, that in the cylindrical region inside the outer cylindrical singularity the particle would be inevitably eaten up by a linear attractive singularity at the z-axis.

The above discussion of the radial geodesic motion is, of course, far from being exhaustive. Its aim was to summarize the physical interpretation of the generated spacetimes and to emphasize the most essential points. The examples of geodesics do not include the most important cases supporting the traditional interpretation — the fall into an attractive singularity at $r = 0$, though it is a typical case for $0 < \sigma < 1$ and may be considered as trivial. The physical qualities of those spacetimes are determined mostly by the character of the seed metric. This conclusion is in full accordance with the principles of the H-M conjecture: all the generated E-M fields represent a generalization of the seed metric that must be their limiting case for a zero electromagnetic field.

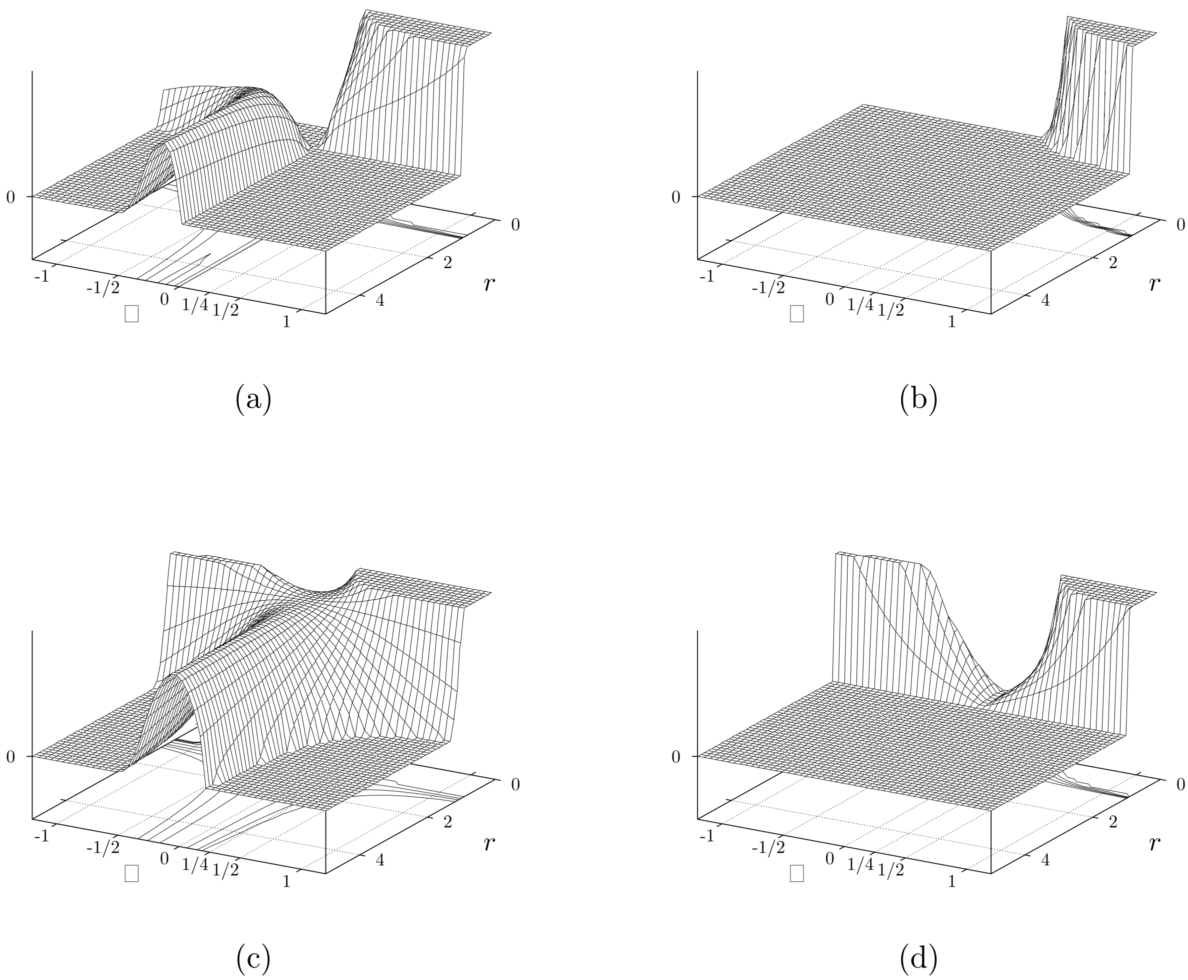


Fig. 3.2: Absolute value of radial velocity for the solution with azimuthal magnetic field. (a) $u_0 = 1,\ q = 0.1$. (b) $u_0 = 1,\ q = 1$. (c) $u_0 = 2,\ q = 0.1$. (d) $u_0 = 2,\ q = 1$.

3.11 Newtonian limit

It is well known that any static axisymmetric vacuum metric with vanishing cosmological constant can be expressed in the Weyl form (see e.g. [3, 6])

$$\mathrm{d}s^2 = -e^{2\mu}\mathrm{d}t^2 + e^{-2\mu}\left[e^{2\nu}\left(\mathrm{d}r^2 + \mathrm{d}z^2\right) + r^2\mathrm{d}\varphi^2\right], \tag{3.17}$$

where $\mu = \mu(r, z),\ \nu = \nu(r, z)$ are functions of r and z only. Here t, r, φ, z represent standard cylindrical coordinates in the sense used in this chapter (see the beginning of the section (3.1) for coordinate ranges). One needs just standard calculus to demonstrate, that one of the vacuum Einstein's equations reduces to Laplace's equation for μ

$$\Delta\mu = \frac{\partial^2\mu}{\partial r^2} + \frac{1}{r}\frac{\partial\mu}{\partial r} + \frac{\partial^2\mu}{\partial z^2} = 0. \tag{3.18}$$

Let us remind that the Laplace equation represents an vacuum equation for the gravitational potential in Newtonian theory of gravity. Consequently, for weak static fields μ can be interpreted as an approximate Newtonian potential of the gravitational field. This

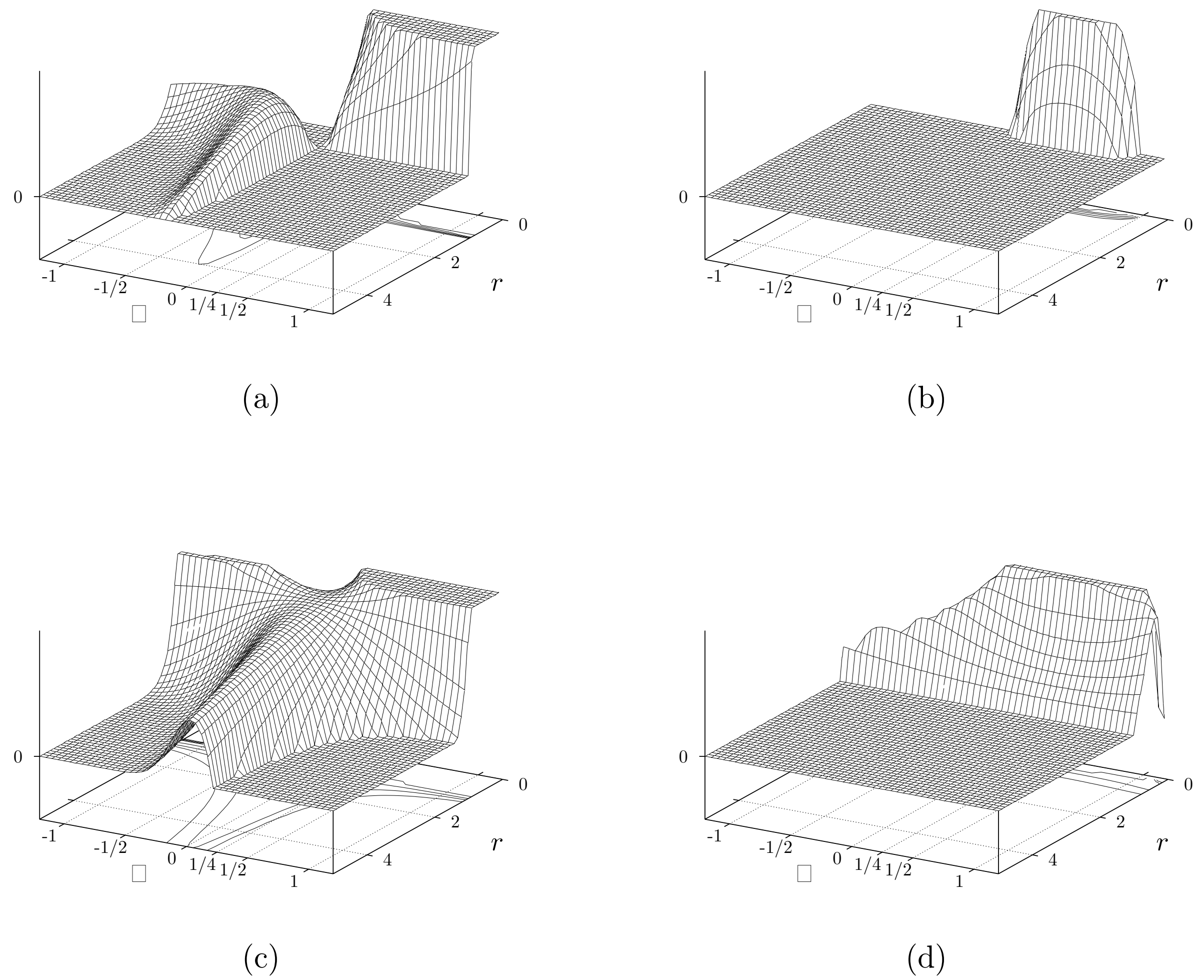


Fig. 3.3: Absolute value of radial velocity for the solution with longitudinal magnetic field. (a) $u_0 = 1,\ q = 0.1$. (b) $u_0 = 1,\ q = 1$. (c) $u_0 = 2,\ q = 0.1$. (d) $u_0 = 2,\ q = 1$.

gives us a guide to the physical meaning of vacuum Weyl solutions, though it should be used with caution, as was illustrated in section (3.1) in the case of the L-C solution.

It is not difficult to show that three of the generated solutions, namely the solution with longitudinal magnetic field (3.9), with longitudinal electric field (section 3.8) and with radial electric field can be expressed in the Weyl form (3.17). One of the Einstein-Maxwell equations takes the form of the Poisson equation

$$\Delta\mu = \frac{\partial^2\mu}{\partial r^2} + \frac{1}{r}\frac{\partial\mu}{\partial r} + \frac{\partial^2\mu}{\partial z^2} = 4\pi\varrho \tag{3.19}$$

for which the right-hand side is determined by the appropriate component of the electromagnetic energy-momentum tensor. Taking into account the well known fact, that Poisson equation represents a non-vacuum equation for the Newtonian gravitational potential. This — in the same sense as in the vacuum case — opens the possibility to extract the Newtonian gravitational potential from the g_{tt} component of the metric tensor, to study it in detail and to compare the results with exact relativistic E-M fields. We shall see, that the qualitative agreement is quite satisfying. On the other hand, the described method is certainly applicable to those E-M fields, that can be put into the Weyl form (3.17) and

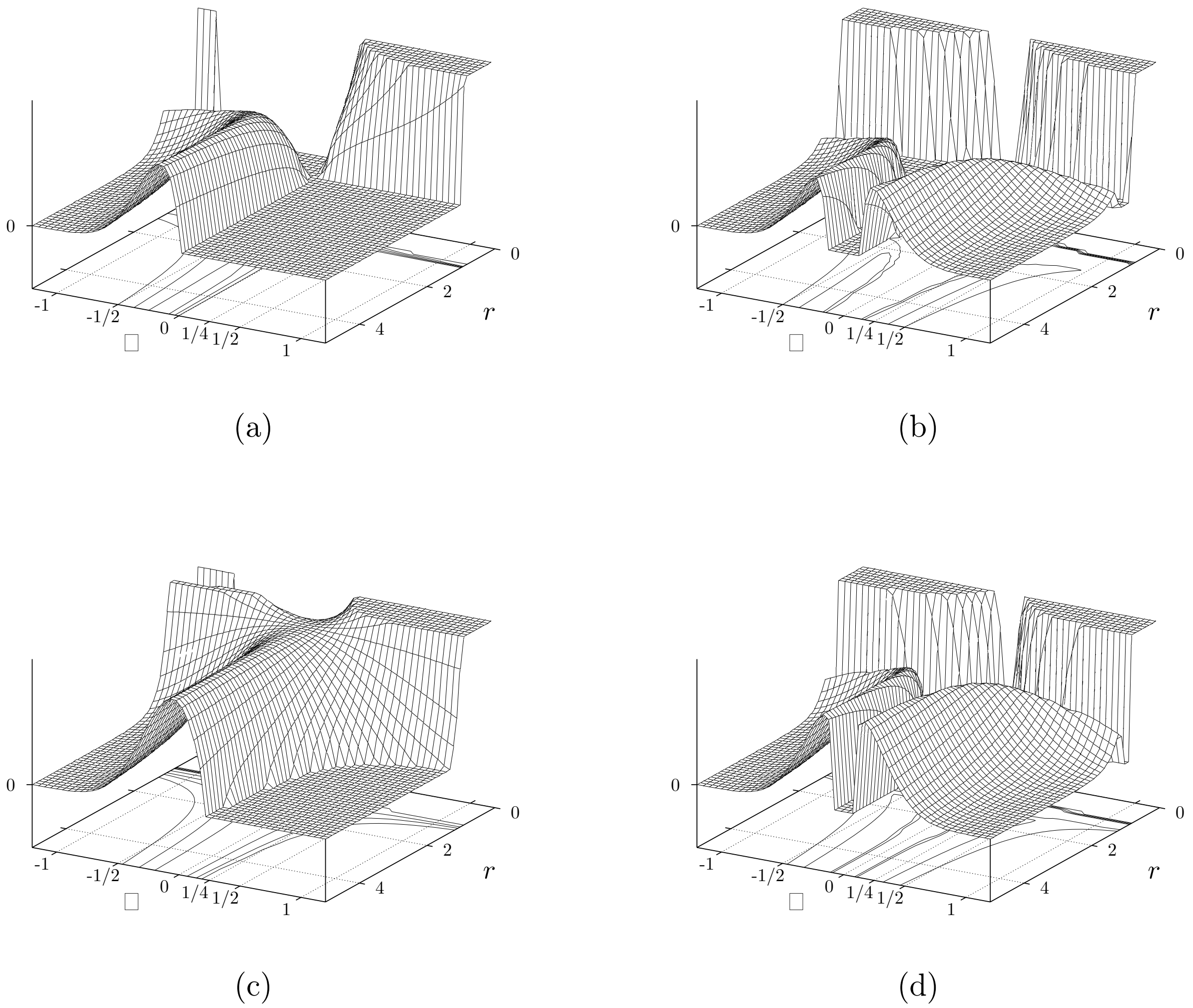


Fig. 3.4: Absolute value of radial velocity for the solution with radial electric field. (a) $u_0 = 1,\ q = 0.1$. (b) $u_0 = 1,\ q = 1$. (c) $u_0 = 2,\ q = 0.1$. (d) $u_0 = 2,\ q = 1$.

gives us no tool how to find Newtonian analogies for the other solutions (3.7), (3.13) and (3.15).

Accordingly, for the solution with a longitudinal magnetic field we obtain Newtonian gravitational potential

$$\phi \equiv \mu = \frac{1}{2}\ln|g_{tt}| = \ln\left[1 + q^2 r^{2(1-2\sigma)}\right] + 2\sigma \ln r.$$

The dependence of ϕ on the radial coordinate r for three different values of σ can be seen in Fig. 3.7(a)–(c). It should be noted, that z-axis is drawn just illustratively (potential ϕ does not depend on z at all) and that in all subplots the cylindrical interpretation of coordinates is preferred, though somebody might reject it for some values of σ. The reason is that the seed L-C solution is usually understood as cylindrically symmetric and the objections against it, no matter how important they might be, are not generally accepted by all authors. The potential well in Fig 3.7(a) qualitatively corresponds to the behaviour of a relativistic particles in Fig. 3.6(a)–(b), the z-axis is protected by a thin potential barrier. Fig. 3.7(b) corresponds to an attractive singularity located along the z-axis. Increasing σ up to the unit value in Fig. 3.7(c) does not lead to a deeper potential

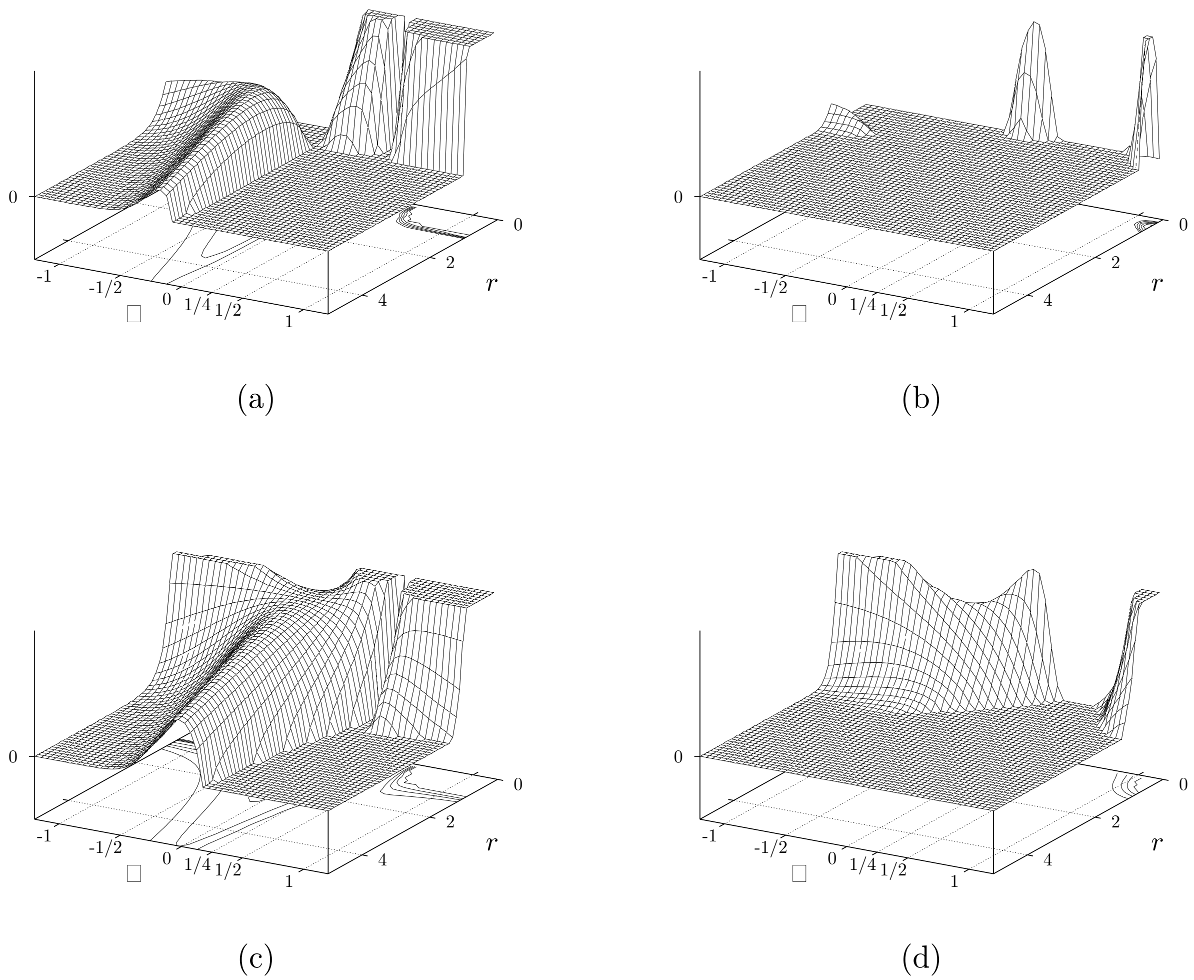


Fig. 3.5: Absolute value of radial velocity for the solution with longitudinal electric field. (a) $u_0 = 1,\ q = 0.1$. (b) $u_0 = 1,\ q = 1$. (c) $u_0 = 2,\ q = 0.1$ (d) $u_0 = 2,\ q = 1$.

well, but a potential bowl which again qualitatively corresponds to the geodesic trajectory in Fig. 3.6(c).

Newtonian potential for the solution with longitudinal electric field described in section 3.8 is endowed with similar features the only exception being the range of values $\sigma \approx 1/2$. Let us recall from the section 3.8 that for $\sigma = 1/2$ the E-M field is flat. Newtonian gravitational potential

$$\phi \equiv \mu = \frac{1}{2} \ln |g_{tt}| = \ln \left[1 + \frac{q^2}{(2\sigma - 1)^2} r^{2(1-2\sigma)} \right] + 2\sigma \ln r$$

diverges for $\sigma = 1/2$ and for close values ($\sigma = 0.49999$ in Fig. 3.7(d)) it is a nearly constant, but non-zero function.

The presence of another cylindrical singularity in the E-M field with radial electric field is also inherited in the shape of the corresponding Newtonian potential

$$\phi \equiv \mu = \frac{1}{2} \ln |g_{tt}| = -\frac{1}{2} \ln \left[\left(1 - q^2 r^{4\sigma} \right)^2 \right] + 2\sigma \ln r$$

in Fig. 3.7(e). While the line-mass singularity along the z-axis is attractive for positive

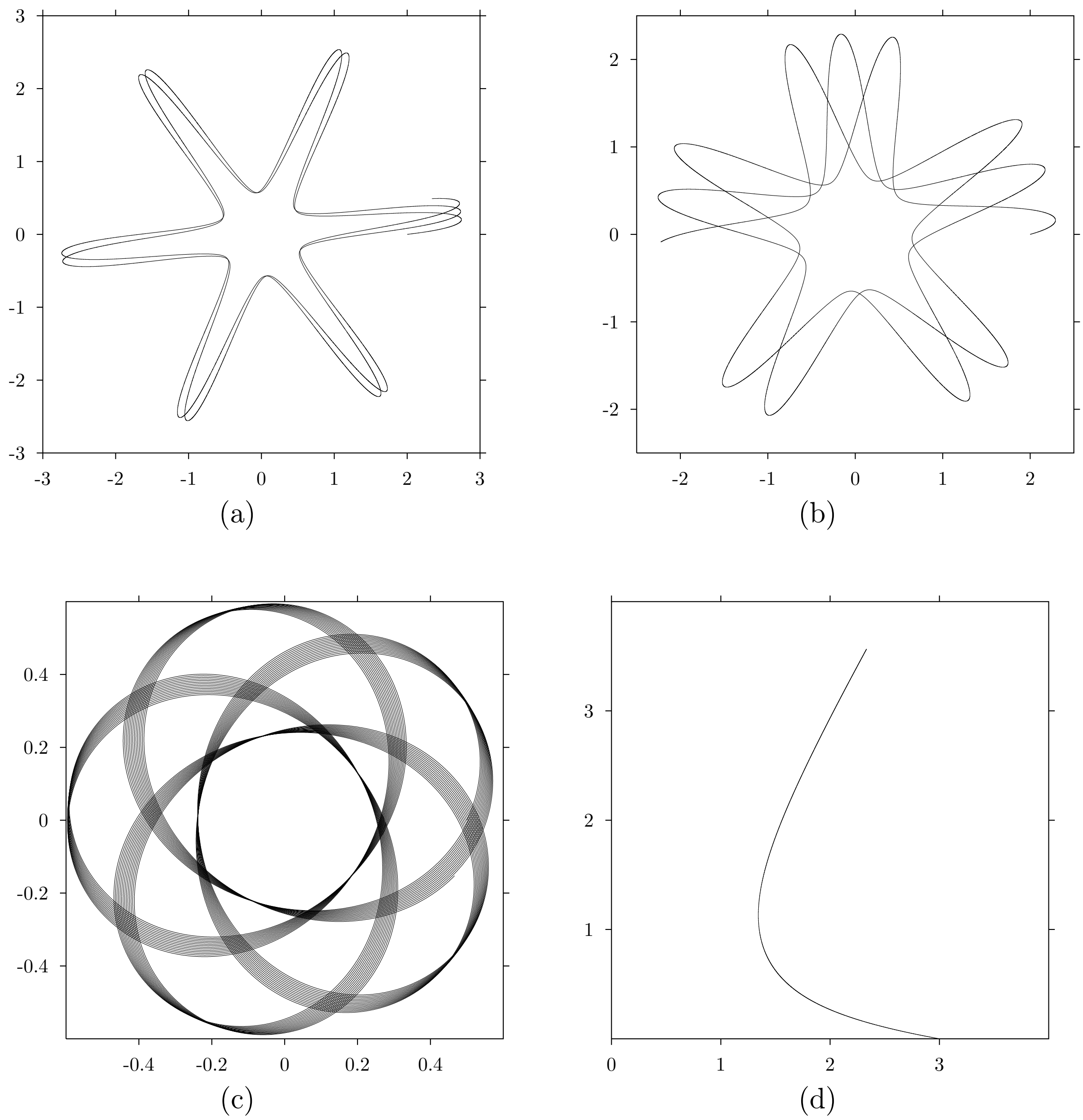


Fig. 3.6: Geodesics on the surfaces of constant z for the L-C solution with longitudinal magnetic field (a) $\sigma = -1/2,\ q = 0.1,\ u_0 = 2,\ u_\varphi = 0.5,\ u_z = 0,\ r_0 = 2$, (b) $\sigma = -1/2,\ q = 0.1,\ u_0 = 1,\ u_\varphi = 0.5,\ u_z = 0,\ r_0 = 2$, (c) $\sigma = 1,\ q = 1,\ u_0 = 2,\ u_\varphi = 0.4,\ u_z = 0,\ r_0 = 0.5$ and for the solution with radial electric field (d) $\sigma = 0.25,\ q = 1,\ u_0 = 2,\ u_\varphi = 1,\ u_z = 0,\ r_0 = 3$.

$\sigma > 0, \sigma \neq 1/2$, the cylindrical singularity repels particles falling into the singularity from outer distant regions. This again comfortably agrees with the relativistic trajectory in Fig 3.6(d).

The comparison of the three discussed E-M fields and their Newtonian analogies can be strengthened by the detailed study of the particle trajectories in the gravitational field of extracted Newtonian potentials. The shapes of the gravitational potential ensure that the Newtonian trajectories should be qualitatively quite similar to those relativistic counterparts in Fig. 3.6. Needless to say that with each Newtonian trajectory three integrals of motion are connected: the energy per unit mass, the angular momentum and the z component of particle's velocity. Here, because of the straightforward correspondence with the relativistic case, they will be again denoted as u_0, u_φ and u_z respectively. Numerical solution of Newton law differential equation then leads to the trajectories drawn in Fig. 3.8; the initial conditions as well as the values of parameters u_0, u_φ, u_z were chosen the same as for the relativistic geodesics lines in Fig. 3.6. The comparison of Figs. 3.6 and 3.8 gives again quite satisfactory qualitative agreement as one should expect after the discussion outlined in the above paragraphs. Glancing at subplots (a),(b) one can see, that relativistic trajectories in Fig. 3.6 are bound closer to the line-mass at the z-axis then the Newtonian ones in Fig. 3.8, which is not surprising again.

3.12 Penrose diagrams

The concept of Penrose diagrams represents nowadays an obligatory part of most standard textbooks, here we follow the notation introduced in [14]. Compactifying and mapping spacetimes onto a finite region by an appropriate transformation is an extremely useful tool to study both conformal and causal structures of the manifold and to understand the character of spacetime singularities.

Here we are interested only in the $t-r$ planes in case of cylindrically symmetric E-M fields of Levi-Civita's type. Due to the cylindrical symmetry these $t-r$ planes remain equivalent along the whole z axis and for the whole range of the azimuthal coordinate φ. Hence the presence of singularities as well as the structure of asymptotically distant regions can be most conveniently demonstrated namely in the $t-r$ planes.

Searching for the null geodesics in this planes had to be done for the E-M field with a radial electric field (3.12) on one side and the metrics (3.1), (3.7), (3.9) on the other side separately.

Most of the "charged" solutions

The Penrose diagrams can be discussed together for the vacuum L-C solution (3.1), for the L-C solutions with an azimuthal magnetic field (3.7), with a longitudinal magnetic field (3.9) and with a longitudinal electric field (section 3.5). For all these solutions we get an identical equation for null geodesics in $t-r$ planes

$$\left|\frac{\mathrm{d}t}{\mathrm{d}r}\right| = \sqrt{-\frac{g_{rr}}{g_{tt}}} = r^{4\sigma(\sigma-1)}. \tag{3.20}$$

Obtained differential equation must be treated for the exponent $4\sigma(\sigma-1) \neq -1$ or $4\sigma(\sigma-1) = -1$, i.e. for $\sigma \neq 1/2$ or $\sigma = 1/2$ separately.

When $\sigma \neq 1/2$ we get the equation of light cones

$$t \pm \frac{r^{4\sigma(\sigma-1)+1}}{4\sigma(\sigma-1)+1} = \text{const.} \tag{3.21}$$

Now mapping the points in infinity into a finite region by means of the inverse tangent function we introduce new coordinates

$$u = \arctan\left[t + \frac{r^{4\sigma(\sigma-1)+1}}{4\sigma(\sigma-1)+1}\right], \quad v = \arctan\left[t - \frac{r^{4\sigma(\sigma-1)+1}}{4\sigma(\sigma-1)+1}\right]. \tag{3.22}$$

Evidently, the equations of line cones determining the causal structure of the spacetime in these new coordinates turn into $u = $ const. and $v = $ const.. Consequently, in Penrose diagrams null geodesics will be represented by lines parallel to quadrant axes (that means containing the angle $\pm\pi/4$ with the horizontal direction). The Penrose diagrams themselves are illustrated in Fig. 3.9(a)–(d). The curves of constant t, r correspond to the values $0,\ \pm 0.25,\ \pm 0.5,\ \pm 1,\ \pm 2$ in all subplots. The symbol i^0 denotes spacelike infinity, the symbols i^+, i^- represent future and past timelike infinity respectively, the symbols $\mathscr{J}^+$, $\mathscr{J}^-$ correspond to future and past null infinity. The character of Penrose diagrams does not depend on the electromagnetic field strength parameter q in case of "charged" spacetimes, hence the Penrose diagrams are relevant not only to the E-M solutions with azimuthal magnetic field, longitudinal magnetic and longitudinal electric field but also for the L-C seed metric. It is not surprising that the conformal structure is very similar to the situation in the Minkowski spacetime (check e.g. Fig. 17.9 in [14]) which is a special case of the seed metric for $\sigma = 0$. The subplots (a)–(d) in Fig. 3.9 differ in the percentage part occupied by regions in certain radial distance in the region covered by the plot. While subplots (a), (c) differ from the Penrose diagrams for the Minkowski spacetime negligibly, in the subplot (b) the most of the plot area represents region in the neighbourhood of the singularity at $r = 0$. On the other hand, for $\sigma = -1/2$ just on the contrary most lines of constant r are jammed close to the singularity at $r = 0$ and the most of the plot represents more distant regions. The latter case is interesting in connection with above discussed geodesic motion and with respect to possible transformation onto a plane-symmetric solution of Taub's type.

If $\sigma = 1/2$ the solution of Eq. (3.20) leads to the light cones

$$t \pm \ln r = \text{const.} \tag{3.23}$$

Performing the same transformation as in the preceding case we get new coordinates

$$u = \arctan\left(t + \ln r\right), \quad v = \arctan\left(t - \ln r\right). \tag{3.24}$$

The resulting Penrose diagram is in Fig. 3.9(e), when one more curve of constant r is added for $r = 4$ compared with the remaining subplots. The features of this conformal diagram are worth exploring in more details. First, let us remind, that all the spacetimes (3.1), (3.7) and (3.9) to which the subplot belongs, become flat for this value of σ and the seed L-C metric have even zero Kretschmann scalar at $r = 0$. The logarithmic function entering the definition of the coordinates u, v in (3.24) has the effect that the line of constant $r = 0$ is mapped into the leftmost point of the graph, from which start all curves of constant time. Moreover, solving the equation (3.23) for in-falling photons one obtains a simple dependence

$$r(t) = A\ \exp(-t),$$

where A is a real constant and thus the z axis at $r = 0$ can be reached in an infinite long time interval. Analogous calculations for the outgoing photons shows that the photon should start at $r = 0$ in an infinite past time coordinate t. That is the main reason why the upper-left and the lower-left border of the plot is also denoted as future and past timelike infinity, though for all other radial distances different form $r = 0$ both of them are located at their "typical" positions at the top and bottom points of the graph. An open question is whether this "defect" could be removed by a suitable transformation.

The solution with radial electric field

Let us turn our attention to the remaining E-M field with a radial electric field described in section 3.5. As it was pointed out in this spacetime we meet one more singularity, which is cylindrical, repulsive and cannot be removed by any transformation of coordinates. Therefore, there is no motivation to perform some special transformation like in the case of Kruskal coordinates. The way we are going to do conformal compactification is quite analogous to that one in the preceding paragraph.

Looking for the equation of the light cones for the metric (3.12) we get

$$\left|\frac{\mathrm{d}t}{\mathrm{d}r}\right| = \sqrt{-\frac{g_{rr}}{g_{tt}}} = \left(1 - q^2 r^{4\sigma}\right)^2 \, r^{4\sigma(\sigma-1)}$$

and expanding the square power then

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left[r^{4\sigma(\sigma-1)} - 2q^2 r^{4\sigma^2} + q^4 r^{4\sigma(\sigma+1)}\right]. \tag{3.25}$$

The first and the third term in square brackets on the right-hand side determine the values of σ that have to be treated separately. Being the solutions of the equations $4\sigma(\sigma\mp 1) = -1$ the values read as $\sigma = \pm 1/2$.

In case $\sigma \neq \pm 1/2$ the equation of null geodesics in the $t - r$ planes has the form

$$t \pm \left[\frac{r^{4\sigma(\sigma-1)+1}}{4\sigma(\sigma-1)+1} - 2q^2\frac{r^{4\sigma^2+1}}{4\sigma^2+1} + q^4\frac{r^{4\sigma(\sigma+1)+1}}{4\sigma(\sigma+1)+1}\right] = \text{const.} \tag{3.26}$$

Consequently, mapping the points at infinity to a finite position by means of arctan function we can introduce coordinates

$$\begin{aligned} u &= \arctan\left[t + \frac{r^{4\sigma(\sigma-1)+1}}{4\sigma(\sigma-1)+1} - 2q^2\frac{r^{4\sigma^2+1}}{4\sigma^2+1} + q^4\frac{r^{4\sigma(\sigma+1)+1}}{4\sigma(\sigma+1)+1}\right], \\ v &= \arctan\left[t - \frac{r^{4\sigma(\sigma-1)+1}}{4\sigma(\sigma-1)+1} - 2q^2\frac{r^{4\sigma^2+1}}{4\sigma^2+1} + q^4\frac{r^{4\sigma(\sigma+1)+1}}{4\sigma(\sigma+1)+1}\right]. \end{aligned} \tag{3.27}$$

The corresponding Penrose diagrams in coordinates u, v are sketched in Fig. 3.10(a)–(d). The character of the diagrams depends now not only on the density parameter σ, but also on the electromagnetic field strength parameter q. The location of the repulsive cylindrical singularity is marked out by a thick curve, the curves of constant t as well as constant r represent the values $0,\ \pm 0.25,\ \pm 0.5,\ \pm 1,\ \pm 2$; one more curve corresponding to $r = 4$ is added.

Analogously, for $\sigma = 1/2$ we obtain

$$\begin{aligned} u &= \arctan\left(t + \ln r - q^2 r^2 + \frac{q^4 r^4}{4}\right), \\ v &= \arctan\left(t - \ln r + q^2 r^2 - \frac{q^4 r^4}{4}\right) \end{aligned} \tag{3.28}$$

and for $\sigma = -1/2$ then

$$\begin{aligned} u &= \arctan\left(t + q^4 \ln r - q^2 r^2 + \frac{r^4}{4}\right), \\ v &= \arctan\left(t - q^4 \ln r + q^2 r^2 - \frac{r^4}{4}\right); \end{aligned} \tag{3.29}$$

corresponding conformal diagrams are in Fig. 3.10(d)–(f). The presence of the logarithm both in (3.28) and (3.29) leads to the difficulties discussed in connection with the subplot 3.9(e). The repulsive cylindrical singularity is again signalized by thick curves.

Summarizing the information learnt from the Penrose diagrams we can see, that all the spacetimes have standard causal structure similar to the causal structure of the seed L-C metric and in some sense even similar to that structure of the Minkowski spacetime described in cylindrical coordinates. Unfortunately, at this state of knowledge we are not able to explain the physical background of the repulsive singularity which appears in the solution with radial electric field (3.12) and to find its classical analogy.

3.13 Survey of results obtained in this chapter

Although some results obtained in this chapter will be further generalized in the following paragraphs, the key concepts, namely the algorithmic scheme formulated in section 3.2 will remain unchanged and can be applied to more general vacuum seed metrics of Weyl's type. Therefore, let us summarize the most interesting outputs of this chapter.

The application of the H-M conjecture to the L-C seed metric revealed several interesting features. First, we have managed to employ all the Killing vectors of the seed vacuum L-C solution in order to obtain electromagnetic fields of both electric and magnetic type through H-M conjecture. The process of generation is marked by common algorithmic steps (though not applicable generally) allowing to devise an instructive scheme (section 3.2) useful even in more complicated calculation in the next chapter. If we admit the standard physical interpretation of the L-C metric acceptable at least for $0 \leq \sigma \leq 1/4$ all obtained solutions describe the E-M fields of an infinite line source ("endless wire") with an electromagnetic field, the source of which can be identified either with a charge or current distribution along the linear source (solutions (3.12) and (3.7) respectively), or with some electromagnetic background (3.9).

Second, we suggest an alternative interpretation of the B-M universe (3.11); it need not necessarily contain a longitudinal magnetic field as is usually supposed. According to the results of section 3.8 the background field might be both electric and magnetic or even their superposition (see section 3.9).

Third, setting $\sigma = -1/2$ and following the transformation found by Bonnor [4] one can reduce each of the generated spacetimes (with the exception (3.15)) to some plane symmetric solution of the Einstein-Maxwell equations. Thus we have obtained also Einstein-Maxwell fields of Taub's type (either electric or magnetic ones) as special cases of found solutions. The explicit form of the metrics line elements is derived in section 4.1.

The analysis of radial geodesic motion in section 3.10 supports the above interpretation of one-dimensional singularity located along the z-axis. The singularity has an *attractive* character that can be explained naturally by the presence of an infinite line source, possible character of which is described in [42, 55] (for a limited range of values of mass density parameter σ only). In comparison with the L-C seed metric the presence of electromagnetic field generally results in a stronger singularity's attraction. Relativistic E-M fields of Levi-Civita's type generate stronger E-M fields than their Newtonian analogies studied in section 3.11.

Finally, the cylindrical symmetric E-M fields 20.9a and 20.9b in [32] do not represent the most general cases since they do not include solutions (3.7), (3.9). In this sense all generated E-M fields with the exception of (3.12) are new. This statement, however, should be understood with caution. The solutions are new in the class of cylindrically symmetric exact solution of Einstein equations written in standard cylindrical coordinates. Unfortunately one cannot exclude the existence of a coordinate transformation that transforms the metrics into other, perhaps already known solutions, written in another coordinate system. This is a classic problem in differential geometry known as the *equivalence problem* (see e.g. [14], p. 178–179). Its classic solution given by Cartan (see e.g. [27] for references) involves computation and comparison of the 10th covariant derivatives of Riemann tensor. Though the invariant classification was enormously improved by works of Karlhede [27], it is still a non-trivial task, best solvable with the help of specially designed software which is not available to the author at this moment.

Namely for this purpose a computer database of exact solution available via Internet was constructed (see page 75 for more details). The long-term goal is to gather all known exact solution in the database and to check any possibly new solution against the database. Answering this equivalence problem for the generated E-M fields is undoubtedly the most urgent task for further work.

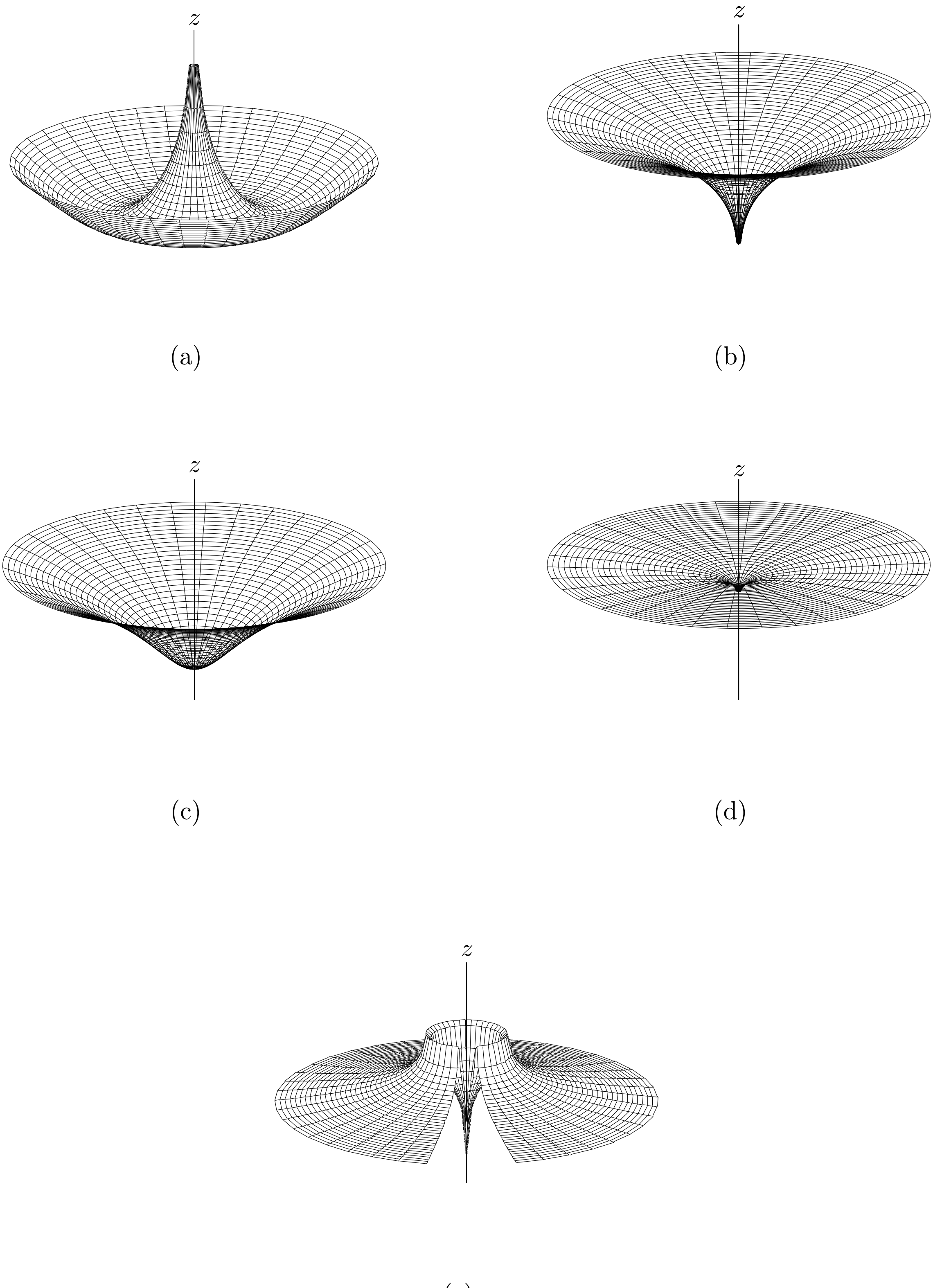


Fig. 3.7: Gravitational potentials for the Newtonian counterpart of the solution with longitudinal magnetic field (a) $\sigma = -1/2,\ q = 0.1$, (b) $\sigma = 1/4,\ q = 1$, (c) $\sigma = 1,\ q = 1$, with longitudinal electric field (d) $\sigma = 0.49999\ q = 1$ and with radial electric field (e) $\sigma = 1,\ q = -1/2$.

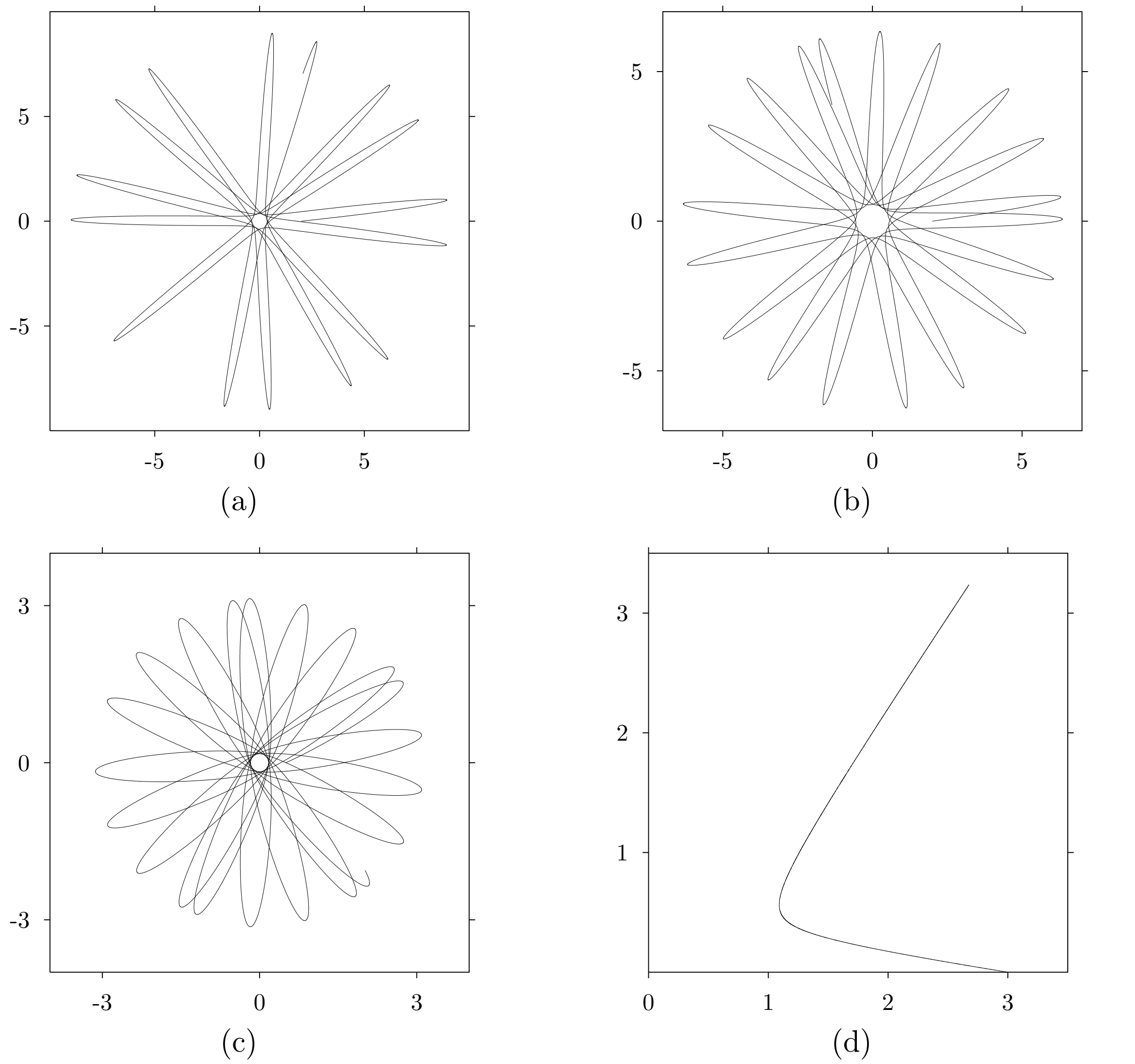


Fig. 3.8: Classical trajectories of free particles in the gravitational field of the Newtonian counterpart in the planes perpendicular to the $z$axis (a) $\sigma = -1/2,\ q = 0.1,\ u_0 = 2,\ u_\varphi = 0.5,\ u_z = 0,\ r_0 = 2$, (b) $\sigma = -1/2,\ q = 0.1,\ u_0 = 1,\ u_\varphi = 0.5,\ u_z = 0,\ r_0 = 2$, (c) $\sigma = 1,\ q = 1,\ u_0 = 2,\ u_\varphi = 0.4,\ u_z = 0,\ r_0 = 0.5$, (d) $\sigma = 0.25,\ q = 1,\ u_0 = 2,\ u_\varphi = 1,\ u_z = 0,\ r_0 = 3$.

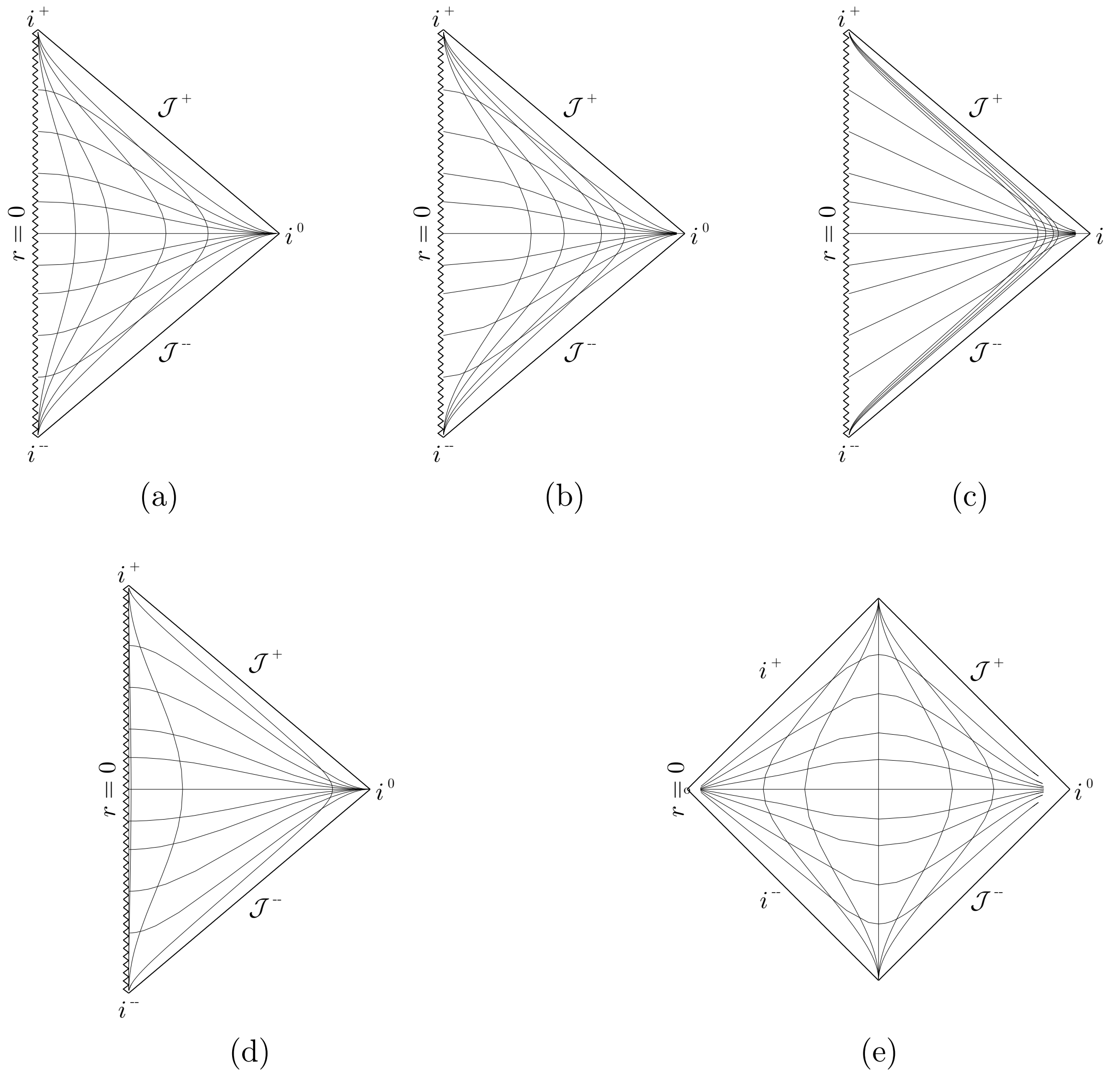


Fig. 3.9: Penrose conformal diagrams of the $r-t$ surfaces for the seed Levi-Civita spacetime, all magnetic solutions and the solution with an longitudinal electric field: (a) $\sigma=0$, (b) $\sigma=0.1$, (c) $\sigma=0.25$, (d) $\sigma=-1/2$, (e) $\sigma=1/2$.

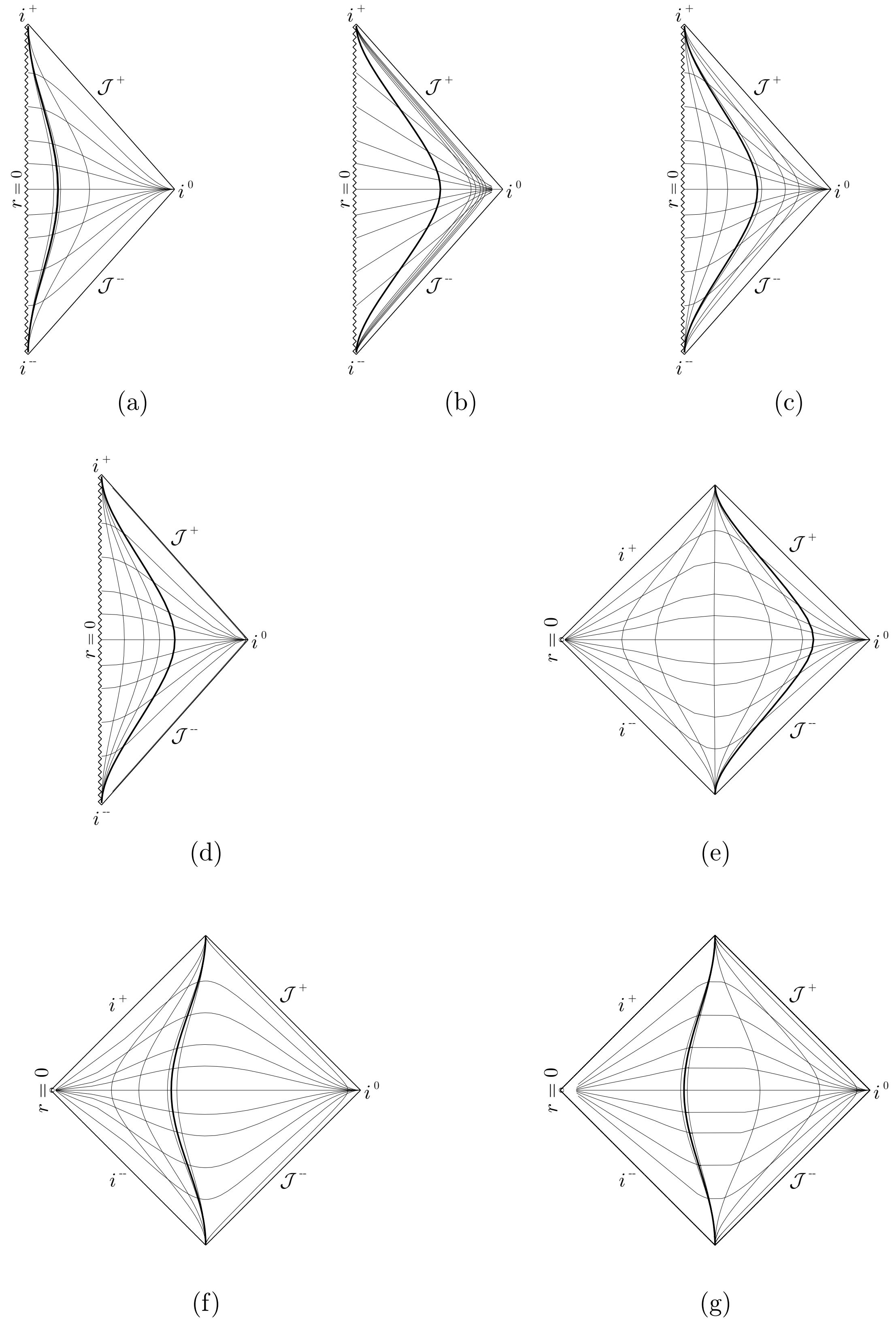


Fig. 3.10: Penrose conformal diagrams of the $r-t$ surfaces for the solution with radial electric field (a) $\sigma = 0.1,\ q = 1$, (b) $\sigma = 0.25,\ q = 0.1$, (c) $\sigma = 1,\ q = 0.1$, (d) $\sigma = 1,\ q = 1$, (e) $\sigma = 1/2,\ q = 0.1$.

Chapter 4

Some other axisymmetric Einstein-Maxwell fields

In this chapter the preceding results are further generalized. In all cases we use the H-M conjecture to generate axisymmetric E-M fields from the static seed metrics of Weyl's type. The algorithmic scheme formulated in section 3.2 is proved to be applicable and useful even in these cases.

4.1 Solutions of Taub's type

As it has been already pointed out in connection with the vacuum L-C solution (section 3.1), Bonnor [4, 6] showed that in the special case $\sigma = -1/2$ the L-C is isometric to the Taub metric

$$\mathrm{d}s^2 = \frac{1}{\sqrt{\xi}}\left(-\mathrm{d}\tau^2 + \mathrm{d}\xi^2\right) + \xi\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{4.1}$$

which is usually called the general plane symmetric solution [6]. The appropriate coordinate transformation

$$t = 2^{2/3}\tau, \quad r = 2^{2/3}\xi^{1/4}, \quad \varphi = 2^{-4/3}x, \quad z = 2^{-4/3}y \tag{4.2}$$

was explicitly found by Bonnor [4]. The second equation in (4.2) requires $\xi \geq 0$. To accept the plane symmetric interpretation of (4.1), one has to reject the interpretation of φ as an periodic angular coordinate used for the L-C solution and admit φ to take an arbitrary real value. Then the original L-C metric for $\sigma = -1/2$ is transformed into a half-space bounded by a hypersurface $\xi = 0$. The singularity located in the L-C solution at the z axis is then spread over the boundary plane $\xi = 0$. While in the L-C solution one has to distinguish between azimuthal vectors collinear with ∂_φ and longitudinal vectors collinear with ∂_z, in the Taub spacetime (4.1) both directions become quite equivalent being parallel to the plane $\xi = 0$.

This illustrates a principal difficulty in interpreting spacetime. Should one regard the plane-symmetric or cylindrical symmetric interpretation as more realistic? Or are they both equally valid each in a different context? For some metric it is possible to choose the coordinate system best suited for the solution. As an example we may take a Schwarzschild solution known to be spherically symmetric, that can appear as the gravitational field of a rod in Weyl cylindrical coordinates; the latter are regarded as unsuitable to describe the Schwarzschild solution. The interpretation of the studied solution is not so unambiguous. Although the nature of the Killing vectors is in favour of the plane-symmetry (this point of view is preferred e.g. by Wang et al. [55]), Bonnor [4] exploring the motion of test particles presented strong arguments supported the axially symmetric interpretation, namely the gravitational field of an semi-infinite line mass discussed below in section 4.4. Another interesting reinterpretation of the Taub singularity was suggested by Jensen and Kučera [26] who again treat the Taub metric as cylindrically symmetric and found a cylindrical analogy to the Einstein-Rosen bridge as well as a cosmic string in its geometry. Regarding the solution (4.1) as plane symmetric or not we have to take into account that the rest particles are repelled by the singularity at $\xi = 0$ and so the plane should have negative mass.

Taking all these fact into account we can proceed to the next logical step and apply the coordinate transformation (4.2) to the E-M fields generated from the seed L-C in the preceding chapter (with the exception of 3.15 for which only the value $\sigma = 1/4$ is acceptable). Here we explicitly reproduce just the line elements and the vector four-potentials, as all components of other tensor objects one can get easily from the corresponding ones in chapter 3 through the coordinate transformation (4.2). Gradually, we obtain the following metrics:

(a) L-C solution with azimuthal magnetic field (3.7) transforms into

$$\begin{gathered} \mathrm{d}s^2 = \left(1+Q^2\xi\right)^2 \left[\frac{1}{\sqrt{\xi}}\left(-\mathrm{d}\tau^2+\mathrm{d}\xi^2\right)+\xi\,\mathrm{d}x^2\right] + \frac{\xi\,\mathrm{d}y^2}{(1+Q^2\xi)^2} \\ \text{with four-potential} \quad \boldsymbol{A} = \frac{Q\,\xi}{1+Q^2\xi}\,\boldsymbol{dy}, \quad Q = 2^{4/3}\,q; \end{gathered} \tag{4.3}$$

(b) L-C solution with longitudinal magnetic field (3.9) transforms into

$$\begin{gathered} \mathrm{d}s^2 = \left(1+Q^2\xi\right)^2 \left[\frac{1}{\sqrt{\xi}}\left(-\mathrm{d}\tau^2+\mathrm{d}\xi^2\right)+\xi\,\mathrm{d}y^2\right] + \frac{\xi\,\mathrm{d}x^2}{(1+Q^2\xi)^2} \\ \text{with four-potential} \quad \boldsymbol{A} = \frac{Q\,\xi}{1+Q^2\xi}\,\boldsymbol{dx}, \quad Q = 2^{4/3}\,q; \end{gathered} \tag{4.4}$$

(c) L-C solution with radial electric field (3.12) transforms into

$$\begin{gathered} \mathrm{d}s^2 = -\frac{\mathrm{d}\tau^2}{\sqrt{\xi}\left(1-\frac{Q^2}{\sqrt{\xi}}\right)^2} + \left(1-\frac{Q^2}{\sqrt{\xi}}\right)^2 \left[\frac{\mathrm{d}\xi^2}{\sqrt{\xi}}+\xi\left(\mathrm{d}x^2+\mathrm{d}y^2\right)\right] \\ \text{with four-potential} \quad \boldsymbol{A} = -\frac{Q}{\sqrt{\xi}\left(1-\frac{Q^2}{\sqrt{\xi}}\right)}\,\boldsymbol{d\tau}, \quad Q = 2^{-2/3}\,q; \end{gathered} \tag{4.5}$$

(d) L-C solution with longitudinal electric field studied in section 3.4 transforms into

$$\begin{gathered} \mathrm{d}s^2 = \left(1+Q^2\xi\right)^2 \left[\frac{1}{\sqrt{\xi}}\left(-\mathrm{d}\tau^2+\mathrm{d}\xi^2\right)+\xi\,\mathrm{d}y^2\right] + \frac{\xi\,\mathrm{d}x^2}{(1+Q^2\xi)^2} \\ \text{with four-potential} \quad \boldsymbol{A} = \frac{Q}{2}\left(y\,\boldsymbol{d\tau} - \tau\,\boldsymbol{dy}\right), \quad Q = 2^{1/3}\,q; \end{gathered} \tag{4.6}$$

(e) L-C special magnetovacuum solution (3.13) transforms into

$$\mathrm{d}s^2 \quad = \quad \frac{\left[1+Q^2\xi\left(x^2+y^2\right)\right]^2}{\sqrt{\xi}}\left(-\mathrm{d}\tau^2+\mathrm{d}\xi^2\right) + \tag{4.7}$$

$$+\frac{\xi}{x^2+y^2}\left\{\left[1+Q^2\xi\left(x^2+y^2\right)\right]^2(x\,\mathrm{d}x+y\,\mathrm{d}y)^2+\frac{(y\,\mathrm{d}x-x\,\mathrm{d}y)^2}{\left[1+Q^2\xi\left(x^2+y^2\right)\right]^2}\right\}$$

$$\text{with four-potential}\quad \boldsymbol{A}=-\frac{2^{-4/3}\,Q\xi}{1+Q^2\left(x^2+y^2\right)\xi}\left(-y\,\boldsymbol{dx}+x\,\boldsymbol{dy}\right),\quad Q=q.$$

The electromagnetic strength parameter Q included in the above equations is rescaled in comparison with the parameter q used in case of E-M fields of Levi-Civita's type. The reason is obvious: not to complicate the mathematical equation with a number of numerical factors which are quite meaningless for our qualitative discussion.

The solutions (4.3) and (4.4) become now quite equivalent, as they both describe E-M field with a magnetic field parallel to the plane $\xi = 0$. Similarly, the spacetime (4.6) represents an electric field parallel to the same plane. The solution (4.8) then corresponds to the general magnetic field with components both parallel and perpendicular to the plane $\xi = 0$. All these solution belong to the Petrov class I.

The most interesting case is probably the E-M field (4.5) containing an electric field perpendicular to the plane $\xi = 0$, the only solution that belongs to the Petrov type D. It could be intuitively interpreted as a gravitational field of a charged plate located at $\xi = 0$. Such solution is known for a long time as McVitie's metric [32], Eq. (13.26) and was even studied in connection with the H-M conjecture by Horský and Mitskievitch [22], by Horský and Novotný [23] and by Fikar and Horský [18]. It is quite natural to expect that our solution (4.5) is isometric with the McVitie's one. Surprisingly, there is no obvious transformation at our disposal and so we again encounter the equivalence problem mentioned in section 3.13: the existence of an suitable coordinate transformation should have been proved through a sophisticated calculus design by Karlhede with help of a suitable software. A promising possibility for further calculations might be an application of the algorithmic scheme described in section 3.2 to the non-stationary plane symmetric Kasner solution (e.g. [32], Eq. (13.31) or [23], Eq. (37)), the line element of which can be obtained from the McVittie's by substituting the time coordinate for ξ in the components of the metric tensor. An E-M field of this type with an electric field perpendicular to the plane $\xi = 0$ has been already found [23] by means of the H-M conjecture.

4.2 Solutions of Robinson-Trautman's type

As it was already mentioned in section 3.1, the L-C solution with the density parameter $\sigma = -1/2$ can be also transformed into the form

$$\mathrm{d}s^2=-\frac{2m}{p}\,\mathrm{d}\eta^2-2\mathrm{d}p\,\mathrm{d}\eta+p^2\left(\mathrm{d}x^2+\mathrm{d}y^2\right),\qquad m=\text{const.}\tag{4.8}$$

via coordinate transformation

$$t=\frac{1}{4}\left(\frac{2}{m}\right)^{2/3}\left(4m\eta+p^2\right),\ r=\left(\frac{2}{m}\right)^{1/6}\sqrt{p},\ \varphi=\left(\frac{m}{2}\right)^{1/3}Cx,\ z=\left(\frac{m}{2}\right)^{1/3}y.\tag{4.9}$$

The vacuum metric (4.8) represents a particular case of the class of radiative vacuum solutions discovered by Robinson and Trautman in 1962. Originally the metric (4.8) was considered to describe a gravitational field of a particle on a null line, but later Bonnor cast doubt upon such interpretation, arguing that (4.9) should be better understood as a special case of a semi-infinite line mass solution described below (see [4] and references cited therein). The whole general class of Robinson-Trautman's solutions have attracted attention in the last decade and the main results are summarized e.g. in [3] with rather skeptic conclusion about the cosmological and astrophysical relevance of these solutions. Because of the unambiguous physical interpretation of the metric (4.8) we are not going to dilate upon it wider, we only briefly list the analytic forms of the transformed E-M fields and their four-potentials. Gradually, we obtain:

a) L-C solution with azimuthal magnetic field (3.7) turns into

$$
\begin{gathered}
\mathrm{d}s^2 = -\frac{2m}{p} f(p)^2\,\mathrm{d}\eta^2 - 2f(p)^2\,\mathrm{d}p\,\mathrm{d}\eta + p^2\left[f(p)^2\,\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{f(p)^2}\right] \\
f(p) = 1 + Q^2p^2, \quad Q = \left(\frac{2}{m}\right)^{1/3} q
\end{gathered}
\tag{4.10}
$$

with four-potential $\boldsymbol{A} = \dfrac{Qp^2}{f(p)}\,\boldsymbol{dy}$;

b) L-C solution with longitudinal magnetic field (3.9) turns into

$$
\begin{gathered}
\mathrm{d}s^2 = -\frac{2m}{p} f(p)^2\,\mathrm{d}\eta^2 - 2f(p)^2\,\mathrm{d}p\,\mathrm{d}\eta + p^2\left[\frac{\mathrm{d}x^2}{f(p)^2} + f(p)^2\,\mathrm{d}y^2\right] \\
f(p) = 1 + Q^2p^2, \quad Q = \left(\frac{2}{m}\right)^{1/3} \frac{q}{C}
\end{gathered}
\tag{4.11}
$$

with four-potential $\boldsymbol{A} = \dfrac{Qp^2}{f(p)}\,\boldsymbol{dx}$;

c) L-C solution with radial electric field (3.12) turns into

$$
\begin{aligned}
\mathrm{d}s^2 = \; & -\frac{2m}{pf(p)^2}\,\mathrm{d}\eta^2 - \frac{2}{f(p)^2}\,\mathrm{d}p\,\mathrm{d}\eta + p^2 f(p)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + \\
& + \frac{\mathrm{d}p^2}{f(p)^2}\left[Q^2\left(\frac{2}{m}\right)^{2/3} + \frac{3}{2}\frac{Q^4}{p}\left(\frac{2}{m}\right)^{1/3} - \frac{Q^6}{p^2} + \frac{Q^8}{8p^3}\,(2m)^{1/3}\right] \\
& f(p) = 1 + Q^2\left(\frac{m}{2}\right)^{1/3}\frac{1}{p}, \quad Q = q
\end{aligned}
\tag{4.12}
$$

with four-potential $\boldsymbol{A} = -\dfrac{Q}{(4m)^{1/3} f(p)}\,(2m\,\boldsymbol{d\eta} + p\,\boldsymbol{dp})$;

d) L-C solution with longitudinal electric field studied in section 3.8 turns into

$$
\begin{gathered}
\mathrm{d}s^2 = -\frac{2m}{p} f(p)^2\,\mathrm{d}\eta^2 - 2f(p)^2\,\mathrm{d}p\,\mathrm{d}\eta + p^2\left[\frac{\mathrm{d}x^2}{f(p)^2} + f(p)^2\,\mathrm{d}y^2\right] \\
f(p) = 1 + Q^2p^2, \quad Q = \left(\frac{2}{m}\right)^{1/3} \frac{q}{2}
\end{gathered}
\tag{4.13}
$$

with four-potential $\boldsymbol{A} = \dfrac{Q}{2}\left[4my\,\boldsymbol{d\eta} + 2yp\,\boldsymbol{dp} + (4m\eta + p^2)\,\boldsymbol{dy}\right]$;

e) L-C special magnetovacuum solution (3.13) turns into

$$\begin{aligned} \mathrm{d}s^2 = \; & -\frac{2m}{p} f(p,x,y)^2 \mathrm{d}\eta^2 - 2f(p,x,y)^2 \mathrm{d}p\,\mathrm{d}\eta + \\ & + \frac{p^2}{x^2+y^2}\left[f(p,x,y)^2 \left(x\mathrm{d}x + y\mathrm{d}y\right)^2 + \frac{\left(y\mathrm{d}x - x\mathrm{d}y\right)^2}{f(p,x,y)^2}\right] \\ & f(p,x,y) = 1 + q^2p^2\left(x^2+y^2\right), \quad Q = q \end{aligned} \tag{4.14}$$

with four-potential $\boldsymbol{A} = \dfrac{Qp^2}{f(p,x,y)}\left(-y\,\boldsymbol{dx} + x\,\boldsymbol{dy}\right)$.

Here we again introduce electromagnetic filed strength parameter Q rescaled with respect the parameter q used for the E-M fields of Levi-Civita's type in chapter 3. The reason is to express the line elements as compactly as possible. The solutions (4.10), (4.11), (4.13), (4.14) belong to the Petrov class I, the metric (4.12) to the Petrov class D. Components of other tensor object are again easily obtainable from the corresponding ones in chapter 3 via the transformation (4.9) and therefore they are not reproduced here.

4.3 General solution for an infinite plane

The vacuum Taub solution (4.1) can be obtained not only as a transformed Levi-Civita metric (3.1) for $\sigma = -1/2$ as discussed in section 4.1, but also as a special case of a general metric for an infinite, in general non-uniform plane [6]

$$\mathrm{d}s^2 = -Z^{4\sigma}\mathrm{d}t^2 + \varrho^2 Z^{2-4\sigma}\mathrm{d}\varphi^2 + Z^{8\sigma^2-4\sigma}\left(Z^2+\varrho^2\right)^{1-4\sigma^2}\left(\mathrm{d}\varrho^2 + \mathrm{d}Z^2\right), \tag{4.15}$$

if one again sets $\sigma = -1/2$. The explicit coordinate transformation converting (4.15) into (4.1) is a composition of the transformation (4.2) and a transformation between cylindrical and Cartesian coordinates and can be put into the form

$$t = 2^{2/3}\tau, \quad \varrho = 2^{-4/3}\sqrt{x^2+y^2}, \quad \varphi = \arctan\left(\frac{y}{x}\right), \quad Z = 2^{2/3}\xi^{1/4}.$$

Now, we have proved, that applying the H-M conjecture to the Taub vacuum spacetime (4.1), namely the adopted algorithm outlined in section 3.2, we can obtain new E-M fields. The fact that the metric (4.1) represents a special case of (4.15) opens a question whether it would be possible to follow this algorithm also in the case of a more general spacetime (4.15). No matter how surprising it may be, the answer is positive: the H-M conjecture enables us to "charge" the metric of an infinite non-uniform plane too. It is evident from the spacetime symmetries, that the solution (4.15) has at least two Killing vectors ∂_t and ∂_Z; both of them generate a new E-M field through the H-M conjecture. The author is not quite certain about the existence of another Killing vector, as general Killing equations for (4.15) are rather complex and difficult to solve.

Therefore, having two Killing vectors at our disposal, one can generate two E-M fields discussed below. Here we just summarize basic characteristic of the obtained E-M fields which can be verified by any suitable computer program. The choice of vector potential as well as the supposed analytical form of the metric that enter the calculations based on the H-M conjecture are in full agreement with the scheme described in section 3.2 and therefore they are just written down here without any comments and additional explanations; the particular steps of calculations are analogous to the generation of all the E-M fields obtained in chapter 3. Because of the correspondence with the seed vacuum metric (4.15) the cylindrical coordinates are used though somebody need not accept them as an ideal choice in case of plane symmetry. On the other hand, once accepting Bonnor's argument that the metric (4.15) should be better interpreted as a gravitational field of a semi-infinite line mass (see section 4.4 below), we may take the cylindrical coordinates quite convenient.

An infinite plane with an electric field

First, let us take the Killing vector ∂_t which leads to the E-M field

$$\mathrm{d}s^2 = -\frac{Z^{4\sigma}}{f(Z)^2}\,\mathrm{d}t^2 + f(Z)^2\left[\varrho^2 Z^{2-4\sigma}\mathrm{d}\varphi^2 + Z^{8\sigma^2-4\sigma}\left(Z^2+\varrho^2\right)^{1-4\sigma^2}\left(\mathrm{d}\varrho^2+\mathrm{d}Z^2\right)\right], \quad (4.16)$$

where $f(Z) = 1 - q^2 Z^{4\sigma}$ with the vector potential

$$\boldsymbol{A} = \frac{qZ^{4\sigma}}{f(Z)}\,\boldsymbol{dt}. \quad (4.17)$$

If we analogously to chapter 3 introduce an orthonormal basis

$$\boldsymbol{\omega}^{(0)} = \frac{Z^{2\sigma}}{f(Z)}\,\boldsymbol{dt}, \qquad \boldsymbol{\omega}^{(1)} = f(Z)Z^{2\sigma(2\sigma-1)}\sqrt{\left(Z^2+\varrho^2\right)^{1-4\sigma^2}}\,\boldsymbol{d\varrho},$$
$$\boldsymbol{\omega}^{(2)} = f(Z)\varrho Z^{1-2\sigma}\,\boldsymbol{d\varphi}, \qquad \boldsymbol{\omega}^{(3)} = f(Z)Z^{2\sigma(2\sigma-1)}\sqrt{\left(Z^2+\varrho^2\right)^{1-4\sigma^2}}\,\boldsymbol{dZ},$$

we obtain the non-zero components of the Einstein tensor

$$G_{(0)(0)} = G_{(1)(1)} = G_{(2)(2)} = -G_{(3)(3)} = 16\,\frac{\sigma^2 q^2\left(Z^2+r^2\right)^{4\sigma^2-1}}{Z^{2\,(2\sigma-1)^2}f(Z)^4}.$$

The electromagnetic field is evidently of an electric type with the electromagnetic invariant satisfying the inequality

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = -32\,\frac{\sigma^2\,q^2\left(Z^2+r^2\right)^{4\sigma^2-1}}{Z^{2\,(2\sigma-1)^2}f(Z)^4} \le 0$$

and the only non-zero component of the electric field strength

$$F^{(0)(3)} = E^{(3)} = 4\,\frac{q\sigma\,\left(z^2+r^2\right)^{(4\sigma^2-1)/2}}{Z^{(2\sigma-1)^2}f(Z)^2}$$

in the Z-direction. The Kretschmann scalar

$$\begin{aligned}\mathcal{R} = 64\,\sigma^2\,&\frac{\left(Z^2+r^2\right)^{8\sigma^2-3}}{Z^{16\sigma^2-8\sigma+4}f(Z)^8}\Big[\Big(16\,r^2\sigma^4 Z^{16\sigma} + 16\,r^2 Z^{16\sigma}\sigma^2 + 24\,r^2 Z^{16\sigma}\sigma^3 + \\ &+ 3\,Z^{2+16\sigma} + Z^{16\sigma}r^2 + 12\,\sigma^2 Z^{2+16\sigma} + 12\,Z^{2+16\sigma}\sigma + 6\,Z^{16\sigma}r^2\sigma\Big)\,q^8 + \\ &+ \left(48\,\sigma^2 Z^{2+12\sigma} + 48\,r^2 Z^{12\sigma}\sigma^2 + 24\,Z^{2+12\sigma}\sigma + 48\,r^2 Z^{12\sigma}\sigma^3 + 12\,Z^{12\sigma}r^2\sigma\right)\,q^6 + \\ &+ \left(-2\,Z^{8\sigma}r^2 - 6\,Z^{2+8\sigma} + 104\,\sigma^2 Z^{2+8\sigma} + 96\,r^2 Z^{8\sigma}\sigma^2 - 32\,r^2\sigma^4 Z^{8\sigma}\right)\,q^4 + \\ &+ \left(48\,r^2\sigma^2 Z^{4\sigma} - 48\,r^2\sigma^3 Z^{4\sigma} - 24\,Z^{2+4\sigma}\sigma + 48\,\sigma^2 Z^{2+4\sigma} - 12\,Z^{4\sigma}r^2\sigma\right)\,q^2 + \\ &+ r^2 + 3\,Z^2 - 24\,r^2\sigma^3 + 16\,r^2\sigma^4 + 16\,r^2\sigma^2 + 12\,Z^2\sigma^2 - 6\,r^2\sigma - 12\,\sigma Z^2\Big]\end{aligned}$$

signalizes the presence of two plane-line singularities, which is predictable also from the line element (4.16). Both singularities are spread over the hypersurfaces of constant Z; the first is located at $Z=0$ (and is "inherited" from the seed metric (4.15)), the second comes from the minus sign in the $f(Z)$ and therefore is located at $Z=\pm q^{-1/(2\sigma)}$ as two infinite condenser-like desks. Finally, finding the non-zero Weyl scalars

$$\begin{aligned}
\Psi_0 = {}& 2r^2\,\sigma\,\frac{\left(Z^2+r^2\right)^{4\sigma^2-2}\,\left(1+q^2Z^{4\sigma}\right)}{Z^{8\sigma^2}f(Z)^2}\,\Big(2\mathrm{i}\,q^2Z^{4\sigma+1}r+8\,q^2Z^{4\sigma}r^2\sigma^2+\\
&+6\,\sigma q^2\,Z^{4\sigma+2}-8\mathrm{i}\,q^2Z^{4\sigma+1}r\sigma^2+r^2\,q^2Z^{4\sigma}+6\,r^2\sigma q^2Z^{4\sigma}+3\,Z^2q^2Z^{4\sigma}-3\,Z^2+\\
&+6\,Z^2\sigma-r^2+6\,r^2\sigma-2\mathrm{i}\,rZ-8\,r^2\sigma^2+8\mathrm{i}\,rZ\sigma^2\Big)\,,\\
\Psi_2 = {}& -\sigma\,\frac{\left(Z^2+r^2\right)^{4\sigma^2-1}}{Z^2f(Z)^4}\,\Big(Z^{-4\,\sigma(2\sigma-3)}q^4+2\,Z^{-4\,\sigma(2\sigma-3)}\sigma q^4-Z^{-4\,\sigma(2\sigma-1)}+\\
&+2\,Z^{-4\,\sigma(2\sigma-1)}\sigma+4\,Z^{-8\,\sigma(\sigma-1)}\sigma q^2\Big)\,,\\
\Psi_4 = {}& \sigma\,\frac{\left(Z^2+r^2\right)^{4\sigma^2-2}\,\left(1+q^2Z^{4\sigma}\right)}{2r^2Z^{8\sigma^2-8\sigma+4}f(Z)^6}\,\Big(-2\mathrm{i}\,q^2Z^{4\sigma+1}r+8\,q^2Z^{4\sigma}r^2\sigma^2+6\,\sigma q^2Z^{4\sigma+2}+\\
&+8\mathrm{i}\,q^2Z^{4\sigma+1}r\sigma^2+r^2\,q^2Z^{4\sigma}+6\,r^2\sigma q^2Z^{4\sigma}+3\,q^2Z^{4\sigma+2}-3\,Z^2+6\,Z^2\,\sigma-\\
&-r^2+6\,r^2\sigma+2\mathrm{i}\,rZ-8\,r^2\sigma^2-8\mathrm{i}\,rZ\sigma^2\Big)\,.
\end{aligned}$$

we can see, that the E-M field (4.16) belongs generally to the Petrov type I. Remembering Petrov classification of the L-C metric and the E-M fields discussed in chapter 3 one should expect that for some values of the parameter σ the Petrov type should reduce to some algebraically special cases. Indeed, trying the values $\sigma=0,\pm1/2,1,1/4$ for which the Petrov type reduces in case of L-C solution, we learn that for $\sigma=0$ the metric (4.16) degenerates to the Petrov type 0, for $\sigma=\pm1/2$ it belongs to the Petrov class D. Following the interpretation of the vacuum seed metric (4.15) as a gravitational field of an infinite plane, then the "charged" solution (4.16) could correspond to a gravitational field of a charged plane described in rest frame of the plane. This conclusion is intuitively supported by the direction of the electric field strength towards the plane or outward from the plane which depends on the sign of the parameter q. No matter how convincing this might seem, it does not explain the physical character of the plane-like singularities.

An infinite plane with an magnetic field

Now let us employ the second Killing vector of the seed vacuum solution (4.15) ∂_φ. The line element

$$\begin{aligned}
\mathrm{d}s^2 = {}& -Z^{4\sigma}f(Z,\varrho)^2\,\mathrm{d}t^2+\frac{\varrho^2Z^{2-4\sigma}}{f(Z,\varrho)^2}\,\mathrm{d}\varphi^2+\\
&+f(Z,\varrho)^2Z^{8\sigma^2-4\sigma}\left(Z^2+\varrho^2\right)^{1-4\sigma^2}\left(\mathrm{d}\varrho^2+\mathrm{d}Z^2\right),
\end{aligned}\tag{4.18}$$

where

$$f(Z,\varrho)=1+q^2\varrho^2Z^{2-4\sigma},$$

then describes an E-M field with four-potential

$$\boldsymbol{A}=\frac{q\varrho^2Z^{2-4\sigma}}{f(Z,\varrho)}\,\boldsymbol{d\varphi}.\tag{4.19}$$

Choosing the basis tetrad

$$\boldsymbol{\omega}^{(0)} = f(Z)Z^{2\sigma}\,\boldsymbol{dt}, \qquad \boldsymbol{\omega}^{(1)} = f(Z)Z^{2\sigma(2\sigma-1)}\sqrt{\left(Z^2+\varrho^2\right)^{1-4\sigma^2}}\,\boldsymbol{d\varrho},$$
$$\boldsymbol{\omega}^{(2)} = \frac{\varrho Z^{1-2\sigma}}{f(Z)}\,\boldsymbol{d\varphi}, \qquad \boldsymbol{\omega}^{(3)} = f(Z)Z^{2\sigma(2\sigma-1)}\sqrt{\left(Z^2+\varrho^2\right)^{1-4\sigma^2}}\,\boldsymbol{dZ}$$

we get the non-zero components of the Einstein tensor

$$G_{(0)(0)} = G_{(2)(2)} = 4q^2\,\frac{\left[\varrho^2\left(2\sigma-1\right)^2+Z^2\right]\left(Z^2+\varrho^2\right)^{4\sigma^2-1}}{Z^{8\sigma^2}f(Z,\varrho)^4},$$
$$G_{(1)(1)} = -\,G_{(3)(3)} = -\,4q^2\,\frac{\left[\varrho^2\left(2\sigma-1\right)^2-Z^2\right]\left(Z^2+\varrho^2\right)^{4\sigma^2-1}}{Z^{8\sigma^2}f(Z,\varrho)^4},$$
$$G_{(1)(3)} = G_{(3)(1)} = -\,8q^2\,\frac{\left(2\sigma-1\right)\varrho Z\left(Z^2+\varrho^2\right)^{4\sigma^2-1}}{Z^{8\sigma^2}f(Z,\varrho)^4}.$$

The electromagnetic field is of magnetic type as one could expect according to the form of the vector potential (4.19). This can be again demonstrated either by the electromagnetic invariant

$$F_{(\mu)(\nu)}F^{(\mu)(\nu)} = 8\,\frac{q^2\left(Z^2+\varrho^2\right)^{4\sigma^2-1}\left[\varrho^2\left(2\sigma-1\right)^2+Z^2\right]}{Z^{8\sigma^2}f(Z,\varrho)^4} \geq 0$$

or directly through the non-zero components of the magnetic field strength that read as

$$F^{(1)(2)} = B^{(3)} = 2\,\frac{qZ\left(Z^2+\varrho^2\right)^{(4\sigma^2-1)/2}}{Z^{4\sigma^2}f(Z,\varrho)^2},$$
$$F^{(2)(3)} = B^{(1)} = 2\,\frac{q\varrho\left(2\sigma-1\right)\left(Z^2+\varrho^2\right)^{(4\sigma^2-1)/2}}{Z^{4\sigma^2}f(Z,\varrho)^2}.$$

So the magnetic field strength $\boldsymbol{B}$ has both the component parallel or perpendicular to the plane at $Z=0$. While the source of the perpendicular component could be identified with the electric currents in the singular plane $Z=0$, the source of the other component of the magnetic field could be hardly generated by some current distribution in the plane, at least according to our classical intuition.

The Kretschmann scalar

$$\begin{aligned}
\mathcal{R} = 64\,&\frac{\left(Z^2+\varrho^2\right)^{8\sigma^2-3}}{Z^{16\sigma^2+8\sigma+4}f(Z,\varrho)^8}\,\Big[\Big(3\,\varrho^{10}\,Z^8+3\,\varrho^4\,Z^{14}-54\,\varrho^8\,Z^{10}\,\sigma+\\
&+117\,\varrho^8\,Z^{10}\,\sigma^2+79\,\varrho^{10}\,Z^8\,\sigma^2-24\,\varrho^{10}\,Z^8\,\sigma+16\,\varrho^{10}\,Z^8\,\sigma^6-\\
&-72\,\varrho^{10}\,Z^8\,\sigma^5+9\,\varrho^8\,Z^{10}+36\,\varrho^8\,Z^{10}\,\sigma^4-108\,\varrho^8\,Z^{10}\,\sigma^3+12\,\varrho^6\,Z^{12}\,\sigma^2+\\
&+136\,\varrho^{10}\,Z^8\,\sigma^4-18\,\varrho^6\,Z^{12}\,\sigma+9\,\varrho^6\,Z^{12}-138\,\varrho^{10}\,Z^8\,\sigma^3\Big)\,q^8+\\
&+\Big(-6\,Z^{4\sigma+12}\varrho^2+48\,Z^{4\sigma+6}\varrho^8\sigma^5+102\,Z^{4\sigma+8}\varrho^6\sigma-210\,Z^{4\sigma+8}\varrho^6\sigma^2+\\
&+192\,Z^{4\sigma+8}\varrho^6\sigma^3-6\,Z^{4\sigma+6}\varrho^8+228\,Z^{4\sigma+6}\varrho^8\sigma^3-18\,Z^{4\sigma+10}\varrho^4-36\,Z^{4\sigma+10}\varrho^4\sigma^2+\\
&+48\,Z^{4\sigma+6}\varrho^8\sigma-72\,Z^{4\sigma+8}\varrho^6\sigma^4+42\,Z^{4\sigma+10}\varrho^4\sigma-18\,Z^{4\sigma+8}\varrho^6-\\
&-168\,Z^{4\sigma+6}\varrho^8\sigma^4-150\,Z^{4\sigma+6}\varrho^8\sigma^2\Big)\,q^6+\Big(-112\,Z^{8\sigma+4}\varrho^6\sigma^3-40\,Z^{8\sigma+4}\varrho^6\sigma-\\
&-136\,Z^{8\sigma+6}\varrho^4\sigma^3+52\,Z^{8\sigma+8}\varrho^2\sigma^2+136\,Z^{8\sigma+6}\varrho^4\sigma^2+5\,Z^{8\sigma+4}\varrho^6\\
&+112\,Z^{8\sigma+4}\varrho^6\sigma^2+5\,Z^{8\sigma+10}+80\,Z^{8\sigma+6}\varrho^4\sigma^4-46\,Z^{8\sigma+8}\varrho^2\sigma-24\,Z^{8\sigma+4}\varrho^6\sigma^4+
\end{aligned}$$

$$
\begin{aligned}
&+ 15\, Z^{8\sigma+6}\varrho^4 + 15\, Z^{8\sigma+8}\varrho^2 - 32\, Z^{8\sigma+4}\varrho^6\sigma^6 - 74\, Z^{8\sigma+6}\varrho^4\sigma + 96\, Z^{8\sigma+4}\varrho^6\sigma^5\Big)\, q^4 + \\
&+ \Big(6\, Z^{12\sigma+2}\varrho^4\sigma^2 - 6\, Z^{\sigma+4}\varrho^2\sigma + 72\, Z^{12\sigma+2}\varrho^4\sigma^4 + 18\, Z^{12\sigma+4}\varrho^2\sigma^2 - 36\, Z^{12\sigma+2}\varrho^4\sigma^3 - \\
&- 24\, Z^{12\sigma+4}\varrho^2\sigma^4 - 12\, Z^{12\sigma+6}\sigma^2 + 6\, Z^{12\sigma+6}\sigma - 48\, Z^{12\sigma+2}\varrho^4\sigma^5\Big)\, q^2 - 6\, Z^{16\sigma}\varrho^2\sigma^3 \\
&+ 12\, Z^{16\sigma+2}\sigma^4 - 24\, Z^{16\sigma}\varrho^2\sigma^5 + 16\, Z^{16\sigma}\varrho^2\sigma^6 + 3\, Z^{16\sigma+2}\sigma^2 + Z^{16\sigma}\varrho^2\sigma^2 \\
&+ 16\, Z^{16\sigma}\varrho^2\sigma^4 - 12\, Z^{16\sigma+2}\sigma^3\ \Big]
\end{aligned}
$$

proves the existence of physical singularity at $Z = 0$. Finally, calculating the Weyl scalars

$$
\begin{aligned}
\Psi_0 &= 2\varrho^2\,\sigma\,\frac{\left(Z^2+\varrho^2\right)^{4\sigma^2-2}\,\left(1+q^2Z^{4\sigma}\right)}{Z^{8\sigma^2}f(Z,\varrho)^2}\,\Big(2\mathrm{i}\, q^2Z^{4\sigma+1}\varrho + 8\, q^2Z^{4\sigma}\varrho^2\sigma^2 + \\
&\quad + 6\,\sigma q^2Z^{4\sigma+2} - 8\mathrm{i}\, q^2Z^{4\sigma+1}\varrho\sigma^2 + \varrho^2q^2Z^{4\sigma} + 6\,\varrho^2\sigma q^2Z^{4\sigma} + 3\, q^2Z^{4\sigma+2} - \\
&\quad - 3\, Z^2 + 6\, Z^2\sigma - \varrho^2 + 6\,\varrho^2\sigma - 2\mathrm{i}\,\varrho Z - 8\,\varrho^2\sigma^2 + 8\mathrm{i}\,\varrho Z\sigma^2\Big)\,, \\
\Psi_2 &= -\,\sigma\,\frac{\left(Z^2+\varrho^2\right)^{4\sigma^2-1}}{Z^2f(Z,\varrho)^4}\,\Big[Z^{-4\sigma(2\sigma-3)}q^4 + 2\, Z^{-4\sigma(2\sigma-3)}\sigma q^4 + 4\, Z^{-8\sigma(-1+\sigma)}\sigma q^2 + \\
&\quad +2\, Z^{-4\sigma(2\sigma-1)}\sigma - Z^{-4\sigma(2\sigma-1)}\Big]\,, \\
\Psi_4 &= \sigma\,\frac{\left(Z^2+\varrho^2\right)^{4\sigma^2-2}\,\left(1+q^2Z^{4\sigma}\right)}{2\varrho^2Z^{8\sigma^2-8\sigma+4}f(Z,\varrho)^6}\,\Big(-2\mathrm{i}\, q^2Z^{4\sigma+1}\varrho + 8\, q^2Z^{4\sigma}\varrho^2\sigma^2 + \\
&\quad + 6\,\sigma q^2Z^{4\sigma+2} + 8\mathrm{i}\, q^2Z^{4\sigma+1}\varrho\sigma^2 + \varrho^2q^2Z^{4\sigma} + 6\,\varrho^2\sigma q^2Z^{4\sigma} + 3\, Z^2q^2Z^{4\sigma} - \\
&\quad -3\, Z^2 + 6\, Z^2\sigma - \varrho^2 + 6\,\varrho^2\sigma + 2\mathrm{i}\,\varrho Z - 8\,\varrho^2\sigma^2 - 8\mathrm{i}\,\varrho Z\sigma^2\Big)
\end{aligned}
$$

we can conclude, that the exact solution (4.18) belongs to the Petrov class I with algebraically special cases $\sigma = 0, 1/2, 1$ that belong to the Petrov type D.

The study of plain-symmetric Einstein and E-M fields has a long tradition in Brno relativistic group (let us remind e.g. [18, 38] or references cited in [1] as an example) and some E-M fields of this type were generated by means of the H-M conjecture [18]. Probably the most comprehensive survey of those solution was given by Amundsen and Grøn [1]. The E-M fields (4.16) and (4.18) generated above are not endowed with the plane symmetry because of the general dependence of the metric tensor components on ϱ-coordinate. Therefore they can be considered as a generalization of the plane-symmetric solutions in case when the singular plane $z = 0$ is identified with a non-uniform physical source.

4.4 Semi-infinite linear source

Physical interpretation of the E-M fields (4.16) and (4.18) could be considered in another context. As was pointed out by Bonnor [5, 6], the vacuum metric of an infinite non-uniform plane (4.15) can be put into the form

$$
\begin{aligned}
&\mathrm{d}s^2 = -\,X^{2\sigma}\mathrm{d}t^2 + X^{-2\sigma}\left[\left(\frac{X}{2R}\right)^{4\sigma^2}\left(\mathrm{d}r^2+\mathrm{d}z^2\right) + r^2\mathrm{d}\varphi^2\right], \\
&X = R + \epsilon(z-z_1), \qquad R = \sqrt{r^2+(z-z_1)^2}, \qquad \epsilon = \pm 1
\end{aligned}
\tag{4.20}
$$

through the coordinate transformation of [5, 6]

$$
r = Z\varrho, \quad 2\epsilon(z-z_1) = Z^2 - \varrho^2, \quad \varphi = \varphi, \quad t = t. \tag{4.21}
$$

The solution (4.20) evidently belongs to the Weyl class of solutions (3.17) and therefore can be interpreted by means of its approximate Newtonian potential $\phi = 1/2 \ln g_{tt} = \sigma \ln X$ (recall section 3.11 for supporting arguments) which represents the Newtonian potential of a semi-infinite line-mass with (Newtonian!) linear density σ located along the z-axis from z_1 to ∞ (for $\epsilon = -1$) or from z_1 to $-\infty$ (for $\epsilon = 1$) [5, 6]. As in the case of the Levi-Civita solution studied in section 3.1 it seems reasonable to assume that (4.20) gives the spacetime of a semi-infinite linear source for small values of the parameter σ. So far no realistic source covering all possible values of σ is known; for instance setting $\sigma = 1/2$ one gets flat, uniformly accelerated metric [4, 6]. This can be better seen when putting $\sigma = 1/2$ in (4.15 rather than in (4.20) which provides

$$\mathrm{d}s^2 = -\, Z^2\, \mathrm{d}t^2 + \mathrm{d}\varrho^2 + \varrho^2\, \mathrm{d}\varphi^2 + \mathrm{d}Z^2.$$

Representing a flat spacetime, this special case $\sigma = 1/2$ can be brought directly to the Minkowski form

$$\mathrm{d}s^2 = -\, \mathrm{d}\tau^2 + \mathrm{d}\eta^2 + \eta^2 \mathrm{d}\phi^2 + \mathrm{d}\zeta^2 \tag{4.22}$$

by the transformation [5]

$$\tanh t = \frac{\tau}{\zeta}, \quad \epsilon\,(z - z_1) = \frac{1}{2}\left(\zeta^2 - \tau^2 - \eta^2\right), \quad r^2 = \eta^2\left(\zeta^2 - \tau^2\right), \quad \varphi = \phi. \tag{4.23}$$

The radial coordinate r is defined only for $\zeta^2 \geq \tau^2$; that is why only half of the Minkowski spacetime (4.22) is covered by the transformation (4.23). Bonnor [5, 6] concludes that he is unable to interpret (4.20) for $\sigma > 1$, but he definitely refuses the possibility that the metric could describe a semi-infinite linear source in that case. For the limiting value $\sigma = 1$ the metric (4.20) acquires an extra arbitrary constant and admits four Killing vectors [5, 6] and thus it represents a considerably different geometry; therefore we exclude this value from further considerations. As was already mentioned above in section 4.1, for $\sigma = -1/2$ the solution (4.20) gives a transform of Taub's plane metric (4.1).

In general, for any Weyl metric (3.17) some parts of the z-axis may still bear what is called a conical singularity unless the regularity condition (see e.g. [5, 6, 12, 44])

$$\lim_{r \to 0} \nu = 0 \tag{4.24}$$

is fulfilled. When Eq. (4.24) holds the radius of a small circle round the z-axis in the plane $z = \text{const.}$ is equal to 2π. A conical singularity is usually interpreted as a stress holding some massive sources in equilibrium against their mutual gravitational attraction or as an additional source the character of which is still not fully understood [5]. Let us remind that the presence of conical singularities is a typical feature of cosmic strings. Particularly, for the semi-infinite linear source, i.e. for the metric (4.20) the Eq. (4.24) can be rewritten in the form

$$\lim_{r \to 0} \left(\frac{X}{2R}\right)^{4\sigma^2} = 1$$

and a similar conditions can be laid down also for the next two metrics mentioned below.

One would naturally expect that a superposition of suitable semi-infinite linear sources could form an infinite linear source, the Newtonian analogy of the Levi-Civita solution (see sections 3.1 and 3.11) as it perfectly works for the Newtonian potentials [5]. Moreover, superposition of two of more semi-infinite line-masses provides a possibility to interpret some other vacuum metric found by Ehlers and Kundt (see [6] and references cited therein).

From our point of view the key question is whether there is possible to apply the H-M conjecture to the trivial Killing vectors ∂_t, ∂_φ of the vacuum metric (4.20) and thus obtain some E-M fields. Because of the transformation (4.21) all that has to be done is to transform the E-M fields (4.16), (4.18) given in the previous section. Thus we obtain two more E-M field equivalent to the solution (4.16), (4.18), but suggesting different physical interpretation at least for some values of the parameter σ. As all tensor components can be easily got by the transformation (4.21) from those listed in the section 4.3, here we reproduce the corresponding line elements and vector potentials only.

Semi-infinite linear source with an electric field

Transforming the metric (4.16) one obtains

$$\begin{gathered} \mathrm{d}s^2 = -\frac{X^{2\sigma}}{f(z,r)^2}\,\mathrm{d}t^2 + f(z,r)^2 X^{-2\sigma}\left[\left(\frac{X}{2R}\right)^{4\sigma^2}\left(\mathrm{d}r^2+\mathrm{d}z^2\right) + r^2\mathrm{d}\varphi^2\right], \\ f(z,r) = 1 - q^2 X^{2\sigma}, \quad \boldsymbol{A} = \frac{qX^{2\sigma}}{f(z,r)}\boldsymbol{dt}, \end{gathered} \tag{4.25}$$

that is an E-M field with combined radial and longitudinal electric fields. The spacetime is again endowed with two singularities. One of them evidently corresponds to a semi-infinite linear source, the other is defined by the condition

$$f(r,z) = 1 - q^2 X^{2\sigma} = 0.$$

The singularity repels geodesic particles and its location can be traced also in the Newtonian gravitational potential ϕ of an corresponding Newtonian analogy. Looking for this Newtonian potential we again use the fact that the metric (4.25) can be expressed in the Weyl form. Therefore it is possible to extract ϕ from the g_{tt} component of the metric tensor in the way described in section 3.11

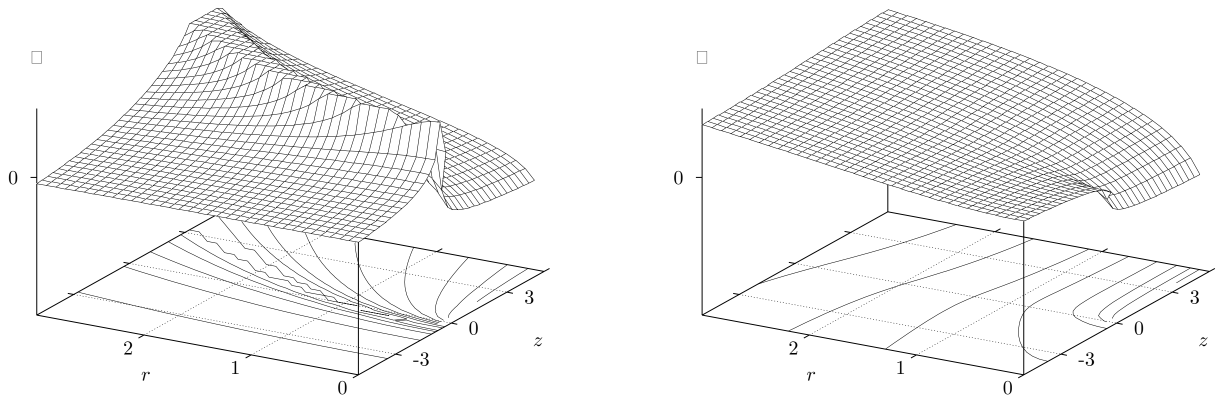
$$\phi = \frac{1}{2}\ln|g_{tt}| = \sigma \ln X - \frac{1}{2}\ln\left[f(r,z)^2\right].$$

The dependence of ϕ on the coordinates r and z together with equipotential curves is sketched in Fig. 4.1(a) where the location of the discussed singularity is marked by the repelling "ridge" dividing the spacetime into two causally separate regions. The attractive singularity lying along the z-axis from $z_1 = 0$ to ∞ represents a semi-infinite linear source.

Semi-infinite linear source with an magnetic field

Transforming the metric (4.18) we come to the magnetovacuum E-M field

$$\begin{gathered} \mathrm{d}s^2 = -f(z,r)^2\left[X^{2\sigma}\mathrm{d}t^2 + X^{-2\sigma}\left(\frac{X}{2R}\right)^{4\sigma^2}\left(\mathrm{d}r^2+\mathrm{d}z^2\right)\right] + \frac{r^2}{X^{2\sigma}f(z,r)^2}\,\mathrm{d}\varphi^2, \\ f(z,r) = 1 + q^2 r^2 X^{-2\sigma}, \quad \boldsymbol{A} = \frac{qr^2X^{-2\sigma}}{f(z,r)}\boldsymbol{d\varphi} \end{gathered} \tag{4.26}$$



(a) (b)

Fig. 4.1: Newtonian potential $\phi = \phi(r, z)$ for the "charged" solutions with $\sigma = 0.25,\ q = 1,\ \epsilon = -1,\ z_1 = 0$: (a) the solution with an electric field; (b) the solution with an magnetic field.

with combined radial and longitudinal background magnetic fields. The metric (4.26) also belongs to the Weyl class which gives us the possibility to extract the corresponding Newtonian potential

$$\phi = \frac{1}{2}\ln|g_{tt}| = \sigma \ln X + \ln\left[f(r,z)\right].$$

drawn in Fig. 4.1(b). The spacetime has only one singularity inherited from the seed metric (4.20) and representing the semi-infinite linear source at the z-axis from $z_1 = 0$ to ∞.

Although the semi-infinite line-mass interpretation of the spacetimes (4.20), (4.25), (4.26) is preferred by some authors to the mathematically equivalent plane symmetric interpretation of the corresponding metrics (4.15), (4.16), (4.18) by Bonnor [5, 6], it also brings serious problems. Let us realize that key arguments supporting the semi-infinite line-mass interpretation come from the behaviour of the approximate Newtonian limit. To accept it definitely we should formulate and fulfil border conditions that one must take into account every time when studying non-infinite sources. This problem is beyond the scope of this work and as far as the author knows in connection with the seed metric (4.20) it has not been treated so far. Without setting realistic border conditions the semi-infinite line-mass interpretation can hardly be regarded as complete.

4.5 C-metric

The C-metric

$$\begin{gathered}\mathrm{d}s^2 = \frac{1}{\mathcal{A}(x+y)^2}\left[\frac{\mathrm{d}y^2}{\mathcal{F}(y)} + \frac{\mathrm{d}x^2}{\mathcal{G}(x)} + \frac{\mathcal{G}(x)\,\mathrm{d}\varphi^2}{K^2} - K^2\mathcal{A}^2\mathcal{F}(y)\,\mathrm{d}t^2\right],\\ \mathcal{F}(y) = -1 + y^2 - 2m\mathcal{A}y^3, \quad \mathcal{G}(x) = 1 - x^2 - 2m\mathcal{A}x^3.\end{gathered} \tag{4.27}$$

was first discovered by Levi-Civita in 1917 and was then further rediscovered several times in various contexts (see e.g. references cited in [12]). It represents one of the Kinnersley's Petrov type D classes [28] with boost-rotation symmetry [44]. The physical interpretation is again rather ambiguous with at least two qualitatively different possibilities neither of which covers the whole range of the coordinates x and y.

Bonnor (see [12, 44] and references cited therein) has showed that some regions of the metric (4.27) can be mapped onto a spacetime describing a gravitational field of a semi-infinite linear source with density parameter $\sigma = 1/2$ lying along the z-axis from $z_1 = 1/(2\mathcal{A})$ to ∞ which was described in section 4.4 (note that one sets $\epsilon = -1$ in (4.20) in that case) or, for another choice of metric parameters, some regions can describe a superposition of finite and infinite linear sources kept apart by the stress represented by a conical singularity. What is even more surprising, although the original C-metric (4.27) is static, it has a time-dependent extension referring to the field of accelerated particles [12, 44]; this possibility was first demonstrated also by Bonnor and this non-stationary extension is connected with the corresponding Weyl's metric through the transformation (4.23).

Although many important and interesting questions arise in connection with the interpretation of the C-metric we are not going to immerse into those details. The fact that C-metric may be considered as an analytic extension of a semi-infinite line-mass metric (4.20) for $\sigma = 1/2$, $z_1 = 1/(2\mathcal{A})$ and $\epsilon = -1$ give us a chance that it could be possible to apply H-M and successfully generate E-M fields. Starting from the original Levi-Civita static form of the line element (4.27) we have two Killing vectors at our disposal: ∂_t, ∂_φ. We shall generate E-M fields corresponding to both Killing vectors. Our task is now much simpler than in all previous cases as one of this E-M is already known and was studied e.g. by Cornish and Uttley [13]. Here we suggest an alternative interpretation to that E-M field: instead of a spacetime with an electric field it can be interpreted as a solution with a magnetic field. Let us now briefly summarize basic characteristics of both E-M fields.

C-metric with an electric field E^y

This metric was studied in detail by Cornish and Uttley [13] who redefined the functions $\mathcal{F}(y)$, $\mathcal{G}(x)$ in the following way

$$\begin{aligned} \mathcal{F}(y) &= -1 + y^2 - 2m\mathcal{A}y^3 + q^2\mathcal{A}^2y^4, \\ \mathcal{G}(x) &= 1 - x^2 - 2m\mathcal{A}x^3 - q^2\mathcal{A}^2x^4 \end{aligned} \tag{4.28}$$

and choose the vector four-potential

$$\boldsymbol{A} = q\mathcal{A}Ky\, \boldsymbol{dt},$$

which leads to the E-M field of an electric type with the electromagnetic invariant

$$F_{\mu\nu}F^{\mu\nu} = -\,2q^2\mathcal{A}^2\,(x+y)^4 \leq 0$$

and the only non-zero component of the electric field (in the coordinate basis)

$$F^{ty} = E^y = \frac{q\mathcal{A}\,(x+y)^4}{K}.$$

Performing the Bonnor's transformation (4.23) Cornish and Uttley prefers the interpretation that this E-M field describes two Reissner-Nordström particles moving with constant proper acceleration [13].

C-metric with an magnetic field B^y

Using H-M conjecture and introducing the four-potential

$$\boldsymbol{A} = \frac{q}{K} x \, \boldsymbol{d\varphi},$$

we can offer another interpretation of the above describe E-M field. Redefining the functions $\mathcal{F}(y)$, $\mathcal{G}(x)$ in accord with Eq. (4.28) and solving E-M equations we get the E-M field of a magnetic type with the electromagnetic invariant

$$F_{\mu\nu} F^{\mu\nu} = 2\mathcal{A}^2 q^2 (x+y)^4 \geq 0$$

and non-zero components of the magnetic field (in the coordinate basis)

$$F^{x\varphi} = -B^y = qK\mathcal{A}^2 \, (x+y)^4 \, .$$

Then, it is also possible to perform the transformation (4.23) and interpret obtained solution as a metric of two non-charged uniformly accelerated particles in some magnetic universe, if one prefers such point of view. However, as it was demonstrated in [12, 13, 44], these suggestions for realistic interpretation of the seed C-metric as well as of the "charged" E-M fields are relevant only for some ranges of coordinates x and y.

4.6 γ-metric

There is an interesting vacuum Weyl metric referring to an isolated mass system, which in some sense generalizes most of the seed vacuum spacetimes discussed in this work (see also next section 4.7). For a special choice of the functions $\mu(r,z)$, $\nu(r,z)$ in (3.17) [21]

$$\begin{aligned} e^{2\mu} &= \left(\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m}\right)^{\gamma} = f_1(r,z), \\ e^{2\nu} &= \left[\frac{(R_1 + R_2 - 2m)(R_1 + R_2 + 2m)}{4R_1 R_2}\right]^{\gamma^2} = f_2(r,z), \\ R_1 &= \sqrt{r^2 + (z-m)^2}, \quad R_2 = \sqrt{r^2 + (z+m)^2}. \end{aligned} \tag{4.29}$$

one gets so called γ-metric, also known as Darmoy-Vorhees-Zipoy solution of vacuum Einstein equations. For our further calculation it appears more convenient to rewrite its line-element in the form

$$\mathrm{d}s^2 = -\,f_1(r,z)\,\mathrm{d}t^2 + f_1(r,z)^{-1}\,\Big[f_2(r,z)\left(\mathrm{d}r^2 + \mathrm{d}z^2\right) + r^2\,\mathrm{d}\varphi\Big]\,. \tag{4.30}$$

The Newtonian image source of the γ-metric corresponds to a finite rod of matter [6, 21] and similarly to previous cases we prefer this interpretation here though it can be also explained as a gravitational field of counterrotating relativistic discs [2, 3]. The reasons for the interpretation based on the Newtonian analogy will be strengthened in the next section because this point of view provides a unifying view at the whole class of spacetimes.

The particular case $\gamma = 1$ corresponds to the Schwarzschild metric outside the Schwarzschild horizon surface. This is more easily seen when we use so-called Erez-Rosen spherical coordinates [20, 21] ϱ, ϑ, φ

$$r^2 = \left(\varrho^2 - 2m\varrho\right)\sin^2\vartheta, \quad z = (\varrho - m)\cos\vartheta, \tag{4.31}$$

which yields the line-element

$$\mathrm{d}s^2 = -\,\mathcal{F}_1(\varrho)\,\mathrm{d}t^2 + \mathcal{F}_1(\varrho)^{-1}\left[\mathcal{F}_2(\varrho,\vartheta)\,\mathrm{d}r^2 + \mathcal{F}_3(\varrho,\vartheta)\,\mathrm{d}\vartheta^2 + \left(\varrho^2 - 2m\varrho\right)\sin^2\vartheta\,\mathrm{d}\varphi\right], \quad (4.32)$$

where

$$\mathcal{F}_1(\varrho) = \left(1 - \frac{2m}{\varrho}\right)^{\gamma}, \quad \mathcal{F}_2(\varrho,\vartheta) = \left(\frac{\varrho^2 - 2m\varrho}{\varrho^2 - 2m\varrho + m^2\sin^2\vartheta}\right)^{\gamma^2-1}, \qquad (4.33)$$
$$\mathcal{F}_3(\varrho,\vartheta) = (\varrho^2 - 2m\varrho)\,\mathcal{F}_2(\varrho,\vartheta).$$

According to [20, 21] and references cited therein the total mass of the γ-metric source is $M = \gamma m$ and its quadrupole moment

$$Q = \frac{\gamma}{3}M^3\left(1 - \gamma^2\right).$$

Thus, the γ-metric expressed in Erez-Rosen coordinates represents a gravitational field of an oblate ($\gamma > 1$) or prolate ($\gamma < 1$) spheroid. Geodesic motion with respect to that interpretation was thoroughly analyzed by Herrera, Paiva and Santos [20] concentrating on deviations from spherical symmetry.

The γ-metric has also an interesting singularity structure: it has a directional singularity for $\gamma > 2$, but not for $\gamma < 2$ [6]. For a distant observer at infinity such gravitational field behaves as an isolated body with monopole and higher mass moments. One possible interior solution was constructed by Steward et al. (check the reference cited in [6]).

The parameters m, γ that enter the expression for the line-element (4.30) are mostly explained in connection with the characteristic of the Newtonian source when we keep standard Weyl cylindrical coordinates. Extracting the Newtonian potential from the g_{tt} component of the metric tensor in the same way as in section 3.11 for the metric (4.30) one obtains

$$\phi = \frac{1}{2}\ln|g_{tt}| = \frac{\gamma}{2}\ln\left(\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m}\right),$$

i.e. a gravitational potential of a linear mass segment of linear mass density $\gamma/2$ and length $2m$ symmetrically distributed along the z-axis [21]. Recently, Herrera et al. showed that extending the length of the rod to infinity, we get the Levi-Civita spacetime [21] (see next section for more details). This statement is not seen at a first glance, as in limit $m \to \infty$ some metric coefficients in (4.30) diverge and the limit is achieved through rather sophisticated Cartan scalar approach. This result once more illustrates the difficulties in the interpretation of both L-C metric (3.1) and γ-metric (4.30), as the notion of a rod prolonged to infinity is relevant for the Newtonian images of the relativistic sources, but it is probably not exact for the relativistic sources themselves because their character is still ambiguous.

From the fact that in some sense the L-C metric (3.1) can be considered as a limiting case of the γ-metric (4.30) a question arises, whether the generating conjecture could be applied to the γ-metric seed vacuum solution in the same way as to the L-C metric, in other words, whether the algorithmic scheme formulated in section 3.2 could work also in this more complicated case. Checking this possibility for two Killing vectors ∂_t, ∂_φ one finds out, that we come to two new E-M fields discussed in the following paragraphs.

γ-metric with an electric field

Let us first employ the timelike Killing vector ∂_t. Analogously to solutions (3.12), (4.5), (4.12), (4.16) and (4.25) we modify the line-element (4.30) into the form

$$\mathrm{d}s^2 = -\frac{f_1(r,z)}{f(r,z)^2}\,\mathrm{d}t^2 + \frac{f(r,z)^2}{f_1(r,z)}\left[f_2(r,z)\left(\mathrm{d}r^2+\mathrm{d}z^2\right)+r^2\,\mathrm{d}\varphi\right], \tag{4.34}$$

where $f(r,z) = 1 - q^2 f_1(r,z)$ and set the four-potential

$$\boldsymbol{A} = q\,\frac{f_1(r,z)}{f(r,z)}\,\boldsymbol{dt}.$$

It can be verified through a standard calculations that sourceless E-M equations with respect to the (4.34) are fulfiled. Choosing the orthonormal tetrad

$$\boldsymbol{\omega}^{(0)} = \frac{\sqrt{f_1(r,z)}}{f(r,z)}\,\boldsymbol{dt}, \qquad \boldsymbol{\omega}^{(1)} = f(r,z)\sqrt{\frac{f_2(r,z)}{f_1(r,z)}}\,\boldsymbol{dr},$$

$$\boldsymbol{\omega}^{(2)} = \frac{r f(r,z)}{\sqrt{f_1(r,z)}}\,\boldsymbol{d\varphi}, \qquad \boldsymbol{\omega}^{(3)} = f(r,z)\sqrt{\frac{f_2(r,z)}{f_1(r,z)}}\,\boldsymbol{dz}$$

for the sake of simplicity, we can gradually express non-zero components of some characteristic tensors. Generated E-M field is of an electric type, because the electromagnetic invariant

$$F_{(\alpha)(\beta)}F^{(\alpha)(\beta)} = -\frac{32\,q^2m^2\gamma^2 f_1(r,z)^2\left\{r^2\left(R_1+R_2\right)^2+\left[z\left(R_1+R_2\right)+m\left(R_1-R_2\right)\right]^2\right\}}{f(r,z)^4 f_2(r,z)R_1^2R_2^2\left(R_1+R_2-2m\right)^2\left(R_1+R_2+2m\right)^2} \leq 0.$$

This can be verified also by calculating the non-zero components of the electric field strength. In Weyl cylindrical coordinates the electric field has both radial and longitudinal components

$$F^{(0)(1)} = E^{(1)} = -\frac{4\,qm\gamma f_1(r,z)r\left(R_1+R_2\right)}{f(r,z)^2\sqrt{f_2(r,z)}R_1R_2\left(R_1+R_2-2m\right)\left(R_1+R_2+2m\right)},$$

$$F^{(0)(3)} = E^{(3)} = -\frac{4\,qm\gamma f_1(r,z)\left[z\left(R_1+R_2\right)+m\left(R_1-R_2\right)\right]}{f(r,z)^2\sqrt{f_2(r,z)}R_1R_2\left(R_1+R_2-2m\right)\left(R_1+R_2+2m\right)}.$$

Standard but rather tedious calculations then provide curvature tensors and scalars. The non-zero tetrad components of the Einstein tensor read as

$$G_{(0)(0)} = G_{(2)(2)} = \frac{32\,q^2m^2\gamma^2 f_1(r,z)^2\left[R_1R_2\left(z^2+r^2-m^2\right)+R_1^2+R_2^2\right]}{f(r,z)^4 f_2(r,z)R_1^2R_2^2\left(R_1+R_2-2m\right)^2\left(R_1+R_2+2m\right)^2}$$

$$G_{(1)(1)} = -G_{(3)(3)} = \frac{32\,q^2m^2\gamma^2 f_1(r,z)^2\left[R_1R_2\left(z^2-r^2-m^2\right)+m^4+z^4-r^4-2z^2m^2\right]}{f(r,z)^4 f_2(r,z)R_1^2R_2^2\left(R_1+R_2-2m\right)^2\left(R_1+R_2+2m\right)^2}$$

$$G_{(1)(3)} = G_{(3)(1)} = \frac{64\,q^2m^2\gamma^2 rz f_1(r,z)^2\left(m^2-z^2-r^2-R_1R_2\right)}{f(r,z)^4 f_2(r,z)R_1^2R_2^2\left(R_1+R_2-2m\right)^2\left(R_1+R_2+2m\right)^2}.$$

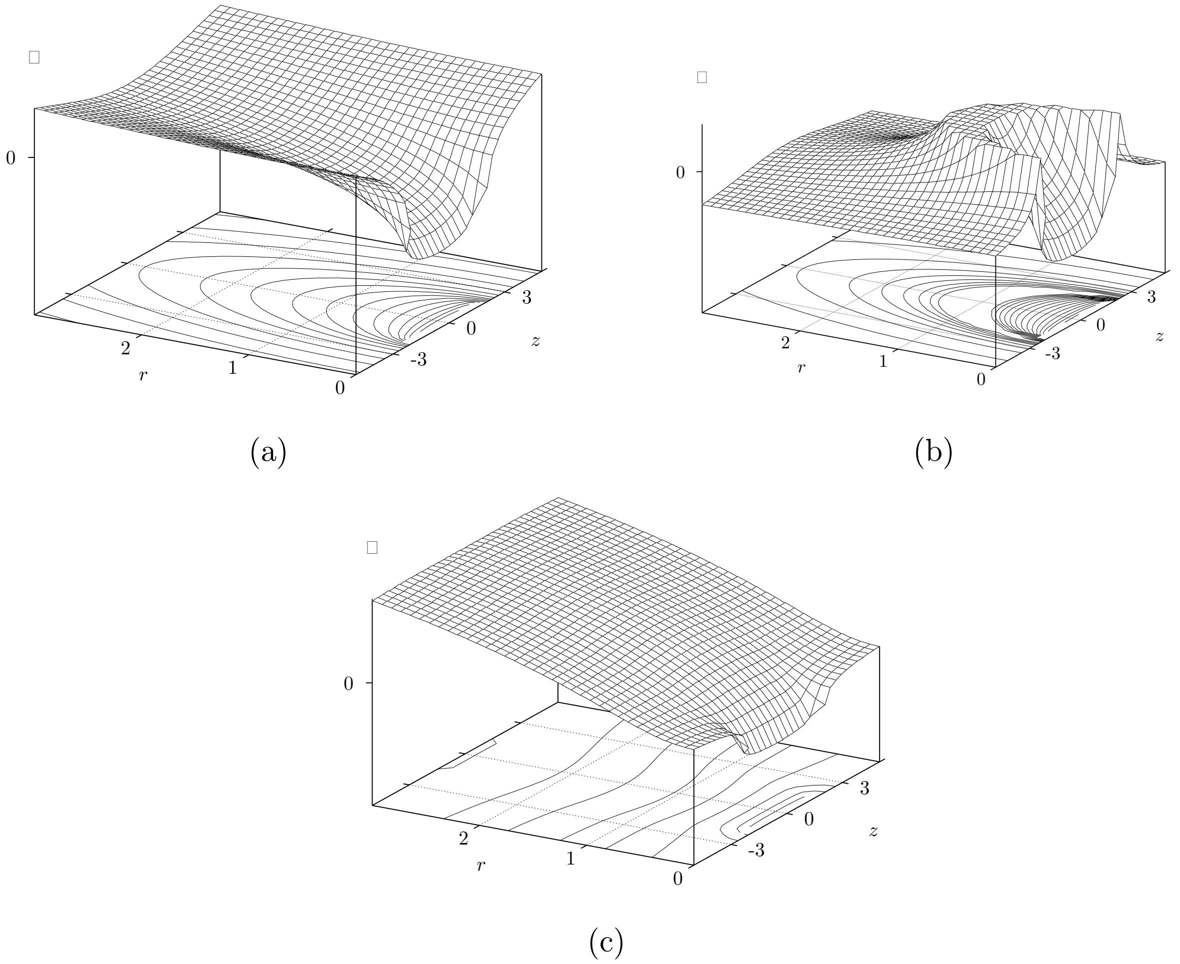


Fig. 4.2: Newtonian potential $\phi = \phi(r, z)$ for the "charged" γ-metric solutions with $m = 2$, $\gamma = 0.5$: (a) the solution with an electric field, $q = 1$; (b) the solution with an electric field, $q = 2$; (c) the solution with an magnetic field, $q = 1$.

The analytic expressions for the Kretschmann scalar as well as for the Weyl scalars are too lengthy to reproduce them here in full extend. The spacetime (4.34) generally belongs to Petrov type I, with the trivial exception $\gamma = 0$ when it belongs to the Petrov type 0. The complicated analytic forms of Weyl scalars makes it very difficult to find other possible special classes algebraically. When substituting "suspicions" values $\gamma = -1, 1/2, 1, 2$ (for the corresponding values of the parameter σ we get special classes for the L-C metric (3.1)), the classification still returns Petrov class I.

Analogously to the E-M fields (3.12), (4.5), (4.12), (4.16), (4.25), the spacetime (4.34) also includes one more singularity at points, the coordinates of which satisfy the equation

$$f(r, z) = 1 - q^2 f_1(r, z) = 1 - q^2 \left(\frac{R_1 + R_2 - 2m}{R_1 + R_2 + 2m} \right)^{\gamma} = 0. \tag{4.35}$$

This transcendent equation can be solved only numerically to localize the singular points. We are going to detect the singularity in the Newtonian gravitational potential as in preceding cases above.

The metric (4.34) is evidently expressed in Weyl form (3.17) which enables us to extract the gravitational potential of the corresponding Newtonian source in the way used several times above in the text (see e.g. section 3.11). This leads to the Newtonian potential

$$\phi = \frac{1}{2}\ln|g_{tt}| = \frac{\gamma}{2}\ln f_1(r,z) - \frac{1}{2}\ln\left[1 - q^2 f_1(r,z)\right]^2,$$

the dependence of which on the coordinates r, z together with the equipotential curves are illustrated in Figs. 4.2(a), (b). While for some set of the parameters the condition (4.35) cannot hold and the spacetime has only one rod-like singularity at z-axis as in Fig. 4.2(a), in other cases the spacetime is endowed with one more singularity surrounding the linear source as a repulsing potential barrier in Fig. 4.2(b). Let us remind that the subplot 4.2(b) is analogous to Fig. 4.2(b) and Fig. 3.7(e). At least part of the electric field may be intuitively connected with the charge distribution along the finite rod, but the physical source of the second singularity present in this solution is not clear.

One might expect that setting $\gamma = 1$ and transforming the metric into Erez-Rosen coordinates (4.31), we could come to the Reissner-Nordström solution. Thus, we again enter the equivalence problem discussed in section 3.13. The components of the metric tensor as well as the components of the electric field do not coincide with those ones for the Reissner-Nordström metric, and so if there exist a coordinate transformation into the Reissner-Nordström solution, it is not obvious.

γ-metric with an magnetic field

Next, turn our attention to the Killing vector ∂_φ. In analogy with the metrics (3.9), (4.4), (4.11), (4.18) and (4.26) we should take the line-element

$$\mathrm{d}s^2 = -f(r,z)^2 f_1(r,z)\,\mathrm{d}t^2 + \frac{1}{f_1(r,z)}\left[f(r,z)^2 f_2(r,z)\left(\mathrm{d}r^2 + \mathrm{d}z^2\right) + \frac{r^2}{f(r,z)^2}\,\mathrm{d}\varphi\right], \quad (4.36)$$

where $f(r,z) = 1 + q^2 r^2/f_1(r,z)$ and four-potential

$$\boldsymbol{A} = q\,\frac{r^2}{f(r,z)f_1(r,z)}\,\boldsymbol{d\varphi}.$$

Working for instance with the basis tetrad

$$\boldsymbol{\omega}^{(0)} = f(r,z)\sqrt{f_1}\,\boldsymbol{dt}, \qquad \boldsymbol{\omega}^{(1)} = f(r,z)\sqrt{\frac{f_2(r,z)}{f_1(r,z)}}\,\boldsymbol{dr},$$

$$\boldsymbol{\omega}^{(2)} = \frac{r}{f(r,z)\sqrt{f_1(r,z)}}\,\boldsymbol{d\varphi}, \qquad \boldsymbol{\omega}^{(3)} = f(r,z)\sqrt{\frac{f_2(r,z)}{f_1(r,z)}}\,\boldsymbol{dz}$$

one can verify the validity of sourceless E-M equations. The electromagnetic field is of magnetic type now because

$$F_{(\alpha)(\beta)}F^{(\alpha)(\beta)} = \frac{32\,q^2}{f(r,z)^4 f_2(r,z) R_1^2 R_2^2 \left(R_1 + R_2 - 2m\right)^2 \left(R_1 + R_2 + 2m\right)^2}\left\{\left[R_1 R_2\left(r^2 + z^2 - m^2\right) - \right.\right.$$
$$\left.\left. - m\gamma r^2\left(R_1 + R_2\right)\right]^2 + r^2 m^2 \gamma^2\left[R_2\left(z - m\right) + R_1\left(z + m\right)\right]^2\right\} \geq 0.$$

The magnetic field has nonzero both longitudinal and radial components

$$F^{(1)(2)} = B^{(3)} = -\frac{4\,q\left[R_1 R_2\left(r^2+z^2-m^2\right)-\,m\gamma r^2\left(R_1+R_2\right)\right]}{f(r,z)^2\sqrt{f_2(r,z)}R_1 R_2\left(R_1+R_2-2m\right)\left(R_1+R_2+2m\right)},$$
$$F^{(2)(3)} = B^{(1)} = -\frac{4\,qm\gamma\left[R_2\left(z-m\right)+R_1\left(z+m\right)\right]}{f(r,z)^2\sqrt{f_2(r,z)}R_1 R_2\left(R_1+R_2-2m\right)\left(R_1+R_2+2m\right)}.$$

Looking for the Einstein tensor we obtain

$$\begin{aligned}
G_{(0)(0)} = G_{(2)(2)} = &\frac{32\,q^2}{f(r,z)^4 f_2(r,z)R_1^2R_2^2\left(R_1+R_2-2m\right)^2\left(R_1+R_2+2m\right)^2}\times\\
&\times\Big[R_1R_2\left(z^2+r^2-m^2\right)\left(m^4+m^2r^2\gamma^2+2r^2m^2-2m^2z^2+r^4+2r^2z^2+z^4\right)+\\
&+2R_1R_2^2m\gamma r^2\left(zm-r^2-z^2\right)-2R_1^2R_2m\gamma r^2\left(zm+r^2+z^2\right)+\\
&+R_1^2R_2^2\left(m^4+2m^2z^2+r^2m^2\gamma^2+z^4+r^4+2r^2z^2\right)\Big],\\
G_{(1)(1)} = -G_{(3)(3)} = &\frac{32\,q^2}{f(r,z)^4 f_2(r,z)R_1^2R_2^2\left(R_1+R_2-2m\right)^2\left(R_1+R_2+2m\right)^2}\times\\
&\times\Big[R_1R_2\left(z^6+r^6-m^6-r^4m^2\gamma^2+3r^4z^2-2r^2z^2m^2-\gamma^2m^2r^2z^2+3r^2z^4+3z^2m^4+\right.\\
&\left.+\gamma^2m^4r^2+r^4m^2-3z^4m^2-r^2m^4\right)+2R_1R_2^2m\gamma r^2\left(zm-r^2-z^2\right)-\\
&-2R_1^2R_2m\gamma r^2\left(zm+r^2+z^2\right)-6m^2r^2z^4+4r^6z^2+2r^6m^2+6r^4z^4+2r^4m^4+\\
&+r^8+z^8+m^8+4r^2z^6+2r^2m^6-4z^6m^2+6z^4m^4-4z^2m^6+r^6\gamma^2m^2-\\
&-\gamma^2m^6r^2-\gamma^2m^2r^2z^4+2\gamma^2m^4r^2z^2\Big],\\
G_{(1)(3)} = G_{(3)(1)} = &\frac{64\,q^2m\gamma r}{f(r,z)^4 f_2(r,z)R_1^2R_2^2\left(R_1+R_2-2m\right)^2\left(R_1+R_2+2m\right)^2}\times\\
&\times\Big[R_1R_2m\gamma r^2-R_1R_2^2\left(r^2z+m^3+z^3-zm-zm^2\right)-\gamma mzr^2\left(m^2-r^2-z^2\right)+\\
&+R_1^2R_2\left(m^3+zm^2-r^2z-z^2m-z^3\right)\Big].
\end{aligned}$$

The Kretschmann scalar and Weyl scalars are again too lengthy to reproduce them here. Generated E-M field belongs to the Petrov type I in general, but for $\gamma = 0$ we get the Petrov type D. Similarly, as in the previous case of the γ-metric with an electric field, for the values $\gamma = -1, 1/2, 1, 2$ that correspond to algebraically special classes of the Levi-Civita metric, the E-M field (4.36) is still of Petrov class I. Unfortunately, we cannot exclude, that the metric could reduce to some algebraically special type for another value of γ. Therefore the complete Petrov classification of both the spacetimes (4.34), (4.36) is still an unclosed problem.

The corresponding Newtonian gravitational potential

$$\phi = \frac{1}{2}\ln|g_{tt}| = \frac{\gamma}{2}\ln f_1\left(r,z\right)+\ln\left[1+\frac{q^2r^2}{f_1\left(r,z\right)}\right],$$

is drawn in Fig. 4.2(c) and has many features in common with the Fig. 4.1(b). This may play an important role when discussing relationships among vacuum Einstein fields of Weyl's type and among corresponding E-M fields generated above. The next section deals with this fields in a slightly more general context. Following the interpretation used throughout this text we may conclude that for small values of γ the spacetime (4.34) describes a field of a finite rod in a magnetic universe.

4.7 Limiting diagram for the studied class of Weyl's metrics

In their inspiring paper [21] Herrera et al. presented an original and fruitful point of view that enables us to systematize our results more efficiently. The unifying idea is based on the comparison of the Newtonian images of the relativistic sources. The relations among the seed vacuum spacetimes are illustrated in Fig. 4.3.

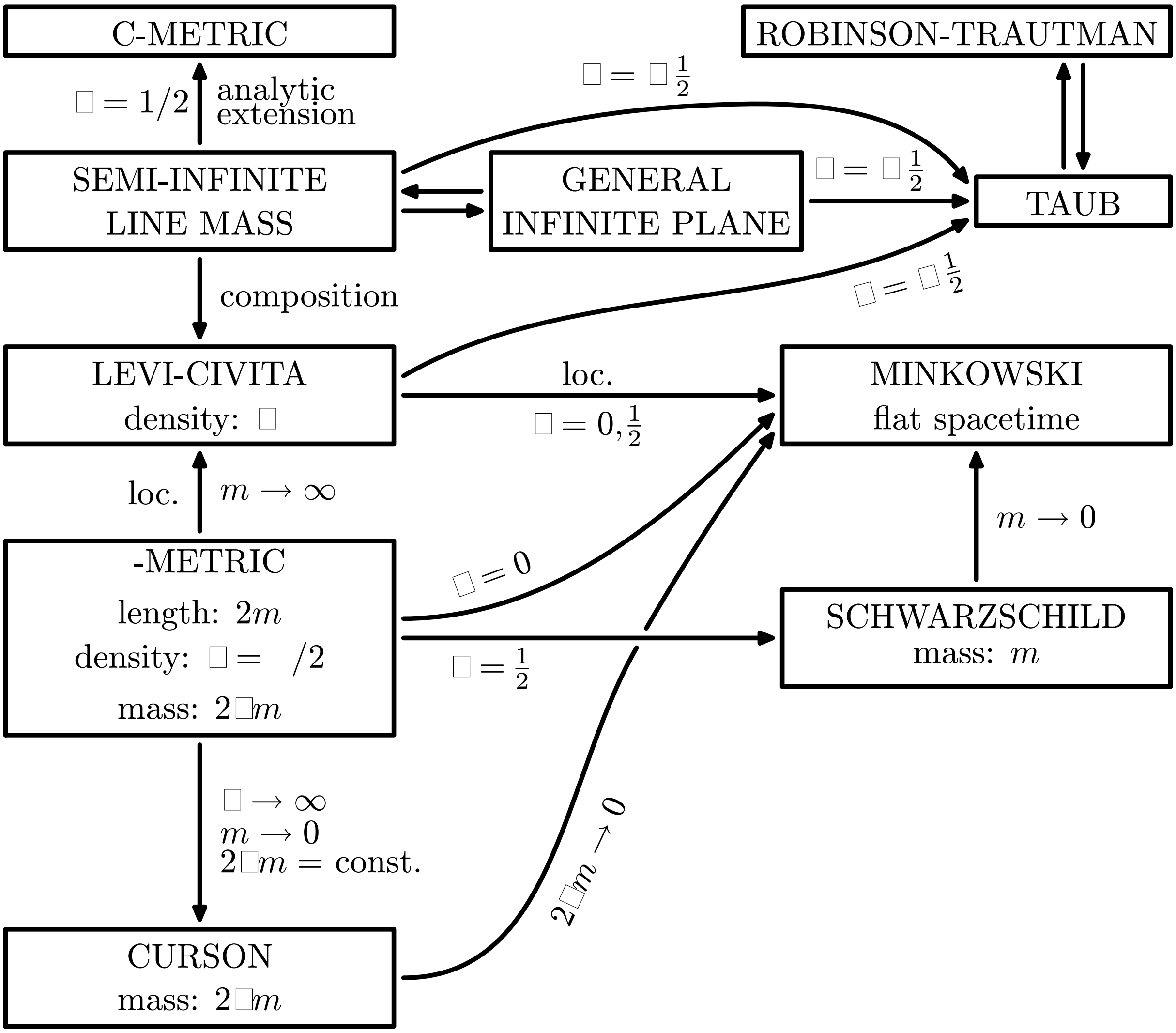


Fig. 4.3: Limiting diagram for the seed vaccum metrics studied in the text (most of the diagram is adapted according to [21]).

In the limiting diagram we have two qualitatively different types of limits. The first type represents an usual limit, when we set a particular value to some metric parameter (or parameters); as an example we can take the Minkowski limit of the γ-metric (4.30) for $\gamma = 0$. The second type of limits is denoted by the abbreviation "loc." in the diagram. In such case the limit is achieved through the Cartan scalars providing a local characterization of the spacetime. Then the limit has to be used with caution, as it does not treat global properties such as topological defects. If, for instance, we set $\sigma = 0$ in the line-element of the L-C metric (3.1), we come to the Minkowski metric written in cylindrical coordinates, but the conical singularity described by the parameter C in the $g_{\varphi\varphi}$ component of the metric tensor survives. Thus the obtained limit differs from the Minkowski spacetime globally.

Another problem connected with the limiting diagram in Fig. 4.3 is a choice of a suitable coordinate system. Following Herrera et al. the cylindrical coordinates of the L-C solution (3.1) are introduced in Fig. 4.3. The coordinates in which the γ-metric takes the form (4.34) are just rescaled L-C coordinates with scaling ratios explicitly given in [21].

Working with the L-C cylindrical coordinates it seems most convenient to express the parameters of other spacetimes through the parameter σ of the L-C solution (3.1); the relations given in Fig. 4.3 are derived in [21]. While handling the limiting relations among the metrics one sometimes has to go through tedious calculations, the physical meaning of that limits looks very transparent when we think about their Newtonian counterparts. Thus, looking for limits we find serious supporting arguments for the interpretation of the studied spacetimes according to the corresponding Newtonian gravitational fields though, as was pointed several times above, in most cases such interpretation is acceptable only for a limited range of the parameters σ or γ.

In that sense, starting from the γ-metric, the Newtonian image of which is a field of a finite rod laid along the z-axis, and prolonging the rod at both ends to $-\infty$ and $+\infty$, i.e. mathematically performing in the limit $m \to \infty$, we get a Newtonian field of an infinite line-mass, the Newtonian analogy of the L-C solution (3.1). Such conclusion sounds quite reasonable for the Newtonian fields, though the matching of the relativistic limits is far from being trivial. If, on the contrary, one contracts the length of the rod to zero at the same time keeping its mass $2\sigma m = \gamma m$ finite we obtain the Curson metric, an interesting space time, the Newtonian potential of which describes a field of a spherical particle. It is again a solution belonging to the Weyl's class (3.17) with function μ and ν in the form [6]

$$\mu = \frac{m}{R}, \quad \nu - \frac{m^2 r^2}{2R^4} \, R = \sqrt{r^2 + z^2}. \tag{4.37}$$

The Curson metric is again endowed with a directional singularity and for a distant observer it looks like a gravitational fields of a point particle with multipoles on it [6]. The Curson solution can also arise as a gravitational field of counterrotating relativistic discs [2, 3]. Another limit of a γ-metric is a well-known Schwarzschild spacetime as was already mentioned in connection with the Erez-Rosen coordinates (4.31) extremely suitable for this limit. As indicated in Fig. 4.3, the γ-metric, Curson metric as well as the Schwarzschild solution reduce to the Minkowski spacetime when the mass of their source falls down to zero. We are not going to discuss the limits in the lower part of the limiting diagram below, because more interesting information about these limits can be found in [21].

Further we would like to concentrate on the upper part which includes the seed vacuum solutions described in this work to which the H-M conjecture was successfully applied. All those limits and coordinate transformations have been already mentioned above in the text, so this chapter provides a brief summary and additional motivation for some steps and considerations we have followed.

An infinite linear source, the Newtonian image of the vacuum L-C solution (3.1), can be considered as a composition of two semi-infinite linear sources, the Newtonian counterparts of the solution (4.20). Following the arguments given in [21], a natural question arises whether the solution (4.20) can also represent a limit of the γ-metric (4.30) analogously to the L-C solution. For the Newtonian images that would mean, that only one end of the finite rod is send to either $+\infty$ or $-\infty$ while the other end is fixed at a finite position. No matter how logical such reasoning could seem, nowadays it is unfortunately speculative and has not been proved. On the other hand, the fact that the H-M conjecture can be applied in the same way to both of the solutions and with analogous results would speak in favour of this possibility. Then, performing the coordinate transformation (4.21), one comes to the field of a non-uniform infinite plane (4.15). The C-metric (4.27) might be understood as an analytic extension of a semi-infinite linear source (4.20) for $\sigma = 1/2$. Finally, the Taub metric (4.1) at the upper-right part of the Fig. 4.3 is a special case of the metrics (3.1), (4.20) and (4.15) for $\sigma = -1/2$ being at the same time isometric with one of the Robinson-Trautman solutions (4.8) through the transformation (4.9).

Although discussed here at the end, the existence of those relations between the seed vacuum metrics was used throughout this chapter. In fact, namely the existence of limits and coordinate transformations justifies the application of the H-M conjecture to all seed vacuum spacetimes in Fig. 4.3 according to the algorithmic scheme formulated in section 3.2. The fact, that obtained E-M fields, both of electric and magnetic type, have an analogical form of the metric tensor and the four-potential of the electromagnetic field, is definitely not a mere coincidence but has its roots in the relations among the seed vacuum spacetimes illustrated in Fig. 4.3. Therefore, the H-M conjecture retrospectively gives a supporting argument to treat all the obtained solutions as particular examples of one general class of metrics. It is hard to judge whether it would be possible to apply the H-M conjecture so effectively and to solve E-M equations in such complicated cases as (4.34) or (4.35), if we had not guessed the form of the metric and vector four-potential from the analogy with simpler cases. It should be pointed out, that the relations among both vacuum and "charged" solutions can be probably best understood through their Newtonian images. That is the key reason why the interpretation of the metrics according to their Newtonian analogies is preferred in this text.

At this point one more question may arise, why this chapter did not start with sections devoted to the γ-metric and why the other spacetimes were not obtained deductively as particular cases. The order, in which the solutions are discussed in this text follows the way how the solutions were gradually generated. The author hopes that such inductive attitude might allow the reader to understand better some problems connected with the application of the H-M conjecture, as it truly mirrors the particular steps of calculations and their motivation reasoning.

Review of obtained results

We have practically proved that H-M conjecture can serve as a useful, efficient tool for the generating of E-M fields. Let us briefly summarize obtained results.

- We present a successful application of the H-M conjecture in its generalized formulation given by Cataldo et al. [7] to the whole class of seed vacuum metrics. In each case all Killing vectors of the seed vacuum solution were used to generate new E-M fields. Although we still do not understand the physical background of the conjecture properly, the results seem to be quite promising.

- An algorithmic scheme for the H-M conjecture formulated in section 3.2 can be used in a considerable amount of various situations. Thus, the scheme represents an interesting contribution to the problematic issues concerning the H-M conjecture and its application.

- We have obtained several new E-M fields generated from the seed vacuum spacetimes of Weyl's type. In chapter 3 there are presented E-M fields of Levi-Civita's type (3.7), (3.9), (3.12), (3.13) and (3.15), then in chapter 4 the E-M fields of Taub's type (4.3)–(4.8), fields of Robinson-Trautman's type (4.10)–(4.14), metrics of "charged" non-uniform infinite plane (4.16),(4.18), metrics of "charged" semi-infinite line-mass (4.25), (4.26), two "charged" C-metrics and finally two "charged" γ-metrics (4.34), (4.36). We have also proved that some of them are connected either via coordinate transformations or via limits when we provide the metric parameters with some special values. It seems to be established that obtained solutions with the exception of (3.12) and the "charged" C-metric are really original when we interpret them in Weyl cylindrical coordinates. There remains an open question whether they cannot be isometric with some other solutions expressed in other coordinate systems and thus considered to have a different type of symmetry.

- One more solution described in section 3.8 is found as a by-product of calculations and a new interpretation is suggested for some already known metrics. For example, in section 3.8 we show that Bonnor-Melvin universe (3.11), usually considered as a magnetovacuum solution, can also represent an electrovacuum solution of E-M equations. On the contrary, the C-metric with an electric field described in section (4.5) and also in [13] can represent magnetic field as well.

- Components of basic tensors are explicitly given and Petrov classes are determined for all metrics. Only in cases of charged γ-metrics (4.34), (4.36) the Petrov classification is perhaps not definite, because the analytic expressions of Weyl scalars are enormously lengthy and complicated to find special solutions algebraically. Particular attention is devoted to the detection and the location of spacetime singularities, though their physical background is not always understood.

- Although it is not the main aim of our considerations, in each case some comments about the physical interpretation of generated spacetimes are added. It is typical for the solutions generated through the H-M conjecture that their interpretation is determined by the physical features of their seed vacuum metrics. As a main source of information we make use of a respectable work done in this field by Bonnor [6]. Unfortunately, for most vacuum metrics discussed above, several possible, often qualitatively different interpretations can be proposed. Following Bonnor's conclusions [5, 6], we found most convenient to interpret both seed vacuum metrics and generated "charged" metrics in accordance with the behaviour of their Newtonian images. At this point we use the possibility that most of the generated spacetimes can be expressed in the Weyl form (3.17) and the Newtonian potential can be easily extracted from the g_{tt} component of the metric tensor (see section 3.11). This attitude is then justified by an impressive qualitative agreement between the behaviour of the geodesic curves and classical trajectories in corresponding Newtonian fields, which is obtained for "charged" solutions of the Levi-Civita type (sections 3.10 and 3.11). In that sense we strengthen Bonnor's arguments in favour of this interpretation, reminding at the same time, that this interpretation is not acceptable for all values of considered parameters. The character of Newtonian gravitational potentials is always demonstrated in illustrative plots providing an intuitive idea about the motion of particles.

As it usually happens in similar situations, most problems, discussed in this work and solved either particularly or to a full extend, induces new questions, that should be answered. Not speaking about principal questions such as theoretical geometrical and physical background of the H-M conjecture, the key problems realized by the author are those ones listed below.

- It is necessary to prove that found solutions of E-M equations are new indeed. This so called equivalence problem is mentioned in section 3.13 and requires a suitable software (SHEEP, CLASSI computer algebra systems [15]) and standard, but rather tedious calculations. The author takes this as his most urgent task for his future work. It might be probably solved within the framework of an international cooperation, e.g. with the group of Professor Skea in Rio de Janeiro (Brazil) or with the group of Professor d'Inverno in Southhampton (Great Britain).

- Petrov classification of the "charged" γ-metric solutions (4.34), (4.36) should be definitively established. At this point SHEEP and CLASSI programs might help again, as MAPLE computer algebra system does not seem to be efficient in such rather complicated cases [15].

- It is necessary to classify obtained E-M fields also with respect to their isometry groups.

- It is necessary investigate the structure of singularities present in the E-M fields of electric type (3.12), (4.5), (4.12), (4.16), (4.25) and (4.34) systematically as it has not been explained in satisfactorily.

- Once preferring the Bonnor's interpretation of some generated spacetimes as gravitational fields of finite, semi-infinite or infinite linear sources, the characteristics of the spacetimes should be completed by calculating the integral quantities such as total electric charge where it is possible. This might again bring new ideas to their physical interpretation. This is relevant for the fields of the electric type, but definitely not for solutions describing material sources in some background electromagnetic field analogous to the B-M universe (3.11).

- More precise insight into the geometrical and physical background of the H-M conjecture might be achieved if we were able to apply the conjecture to some vacuum spacetimes with non-zero cosmological constant Λ, the existence of which nowadays undergoes a revival owing to the observations of distant supernovae with serious cosmological consequences [41]. In connection with our work, especially with the chapter 3, it would be desirable to try the application of the H-M conjecture to the Levi-Civita solution with cosmological constant derived by da Silva et al. [17]. If we were able to find new E-M fields with a non-zero cosmological constant, it would represent an important result, because as far as the author knows, the H-M conjecture has never been used under such conditions.

From the list above it is quite clear this work cannot claim itself to be exhaustive. The author would be happy if it could serve as an inspiring introduction into the problems connected with the H-M generating conjecture and could possibly attract some followers to join this great adventure to search for new exact solutions of E-M equations and for their interpretation.

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Other tools and sources of information

I gratefully acknowledge the possibility to use some computer programs and to exploit various sources of information on the Internet, the most important of which are listed below.

Symbolical, numerical and graphical tools

Algebraic computations were performed with the computer algebra package *Maple V*. Differential equations were solved numerically with the help of the program *Octave*, the graphs were plotted by the package *GNUplot*. Both *Octave* and *GNUplot* are freely available under the conditions of the GNU General Public Licence. For more information please see Internet home-sites:

`http://www.maplesoft.com`
`http://www.che.wisc.edu/octave/`
`http://www.cs.dartmouth.edu/gnuplot_info.html`

GRTensorII and GRTensorM

GRTensorII (for *Maple V*) and GRTensorM (for *Mathematica*) are powerful packages designed for the calculation and the manipulation of tensor components and related objects. They were programmed by *Peter Musgrave, Denis Pollney and Kayll Lake* from the Qeen's University, Kingston, Canada. Both software and documentation [37] can be obtained free of charge from the ftp site at `ftp.astro.queensu.ca` in the `/pub/grtensor` directory, or from the world-wide-web page `http://grtensor.phy.queensu.ca/`.

On-line database of exact solutions

An Internet database of exact solutions is maintained by *Prof. Jim Skea* from the Symbolic Computation Group, Dept. of Theoretical Physics, Instituto de Fisica, Universidade do Estado de Rio de Janeiro, Brazil. The database contains more than 200 spacetimes found by the year 1985, most of which can be found in [32]. It can be reached at the following addresses:

Main site:	`http://edradour.symbcomp.uerj.br` (Brazil.)
Mirrors:	`http://www.astro.queensu.ca/~jimsk` (Kingston, Canada),
	`http://www.maths.soton.ac.uk/~rdi/database/` (Southampton, England).

A few interesting relativity-related Internet places

Working on my thesis I have exploited various pieces of information from www-pages devoted to problems of general relativity. Here is a selection of addresses I have found most useful:

Contacts and links:	http://www.maths.qmw.ac.uk/hyperspace/, http://www.maths.soton.ac.uk/relativity/links.html;
Preprint archives:	http://xxx.lanl.gov/, http://xxx.soton.ac.uk/archive/gr-qc, http://otokar.troja.mff.cuni.cz/veda/gr-qc/, http://www-spires.slac.stanford.edu/find/hep;
Journals:	http://www.lib.cas.cz/knav/journals/eng/ Czechoslovak_Journal_of_Physics.htm (Czech. J. Phys.) http://www.iop.org/Journals/cq/ (Class. Quantum Grav.), http://www.math.uni-potsdam.de/grg/ (Gen. Rel. Grav.). http://www.wkap.nl/journalhome.htm/0001-7701 (Gen. Rel. Grav.) http://prd.aps.org/ (Phys. Rev. D)

Appendix A

Abstract in Czech

Problematiku, kterou se zabývá předložená práce, lze rozdělit do dvou základních okruhů. První z nich představuje generování nových exaktních řešení Einsteinových-Maxwellových rovnic, druhý potom jejich fyzikální interpretace.

Generování prostoročasů s elektromagnetickým polem

Řešení Einsteinových-Maxwellových rovnic reprezentuje obecně nesmírně složitý problém řešitelný přímo pouze v nejjednodušších případech. Na druhé straně nezanedbatelný počet Einsteinových-Maxwellových polí byl získán pomocí speciálních, tzv. generujících technik. Účelem těchto technik je získat nová řešení Einsteinových-Maxwellových rovnic z řešení již známých, popřípadě z vakuových řešení Einsteinových rovnic, v nichž elektromagnetické pole není vůbec přítomno. Tyto výchozí prostoročasy se standardně nazývají „seed“ metrikami, získaná řešení Einsteinových-Maxwellových rovnic se pak zkráceně označují jako „nabitá“. Převážná většina generujících technik různým způsobem využívá izometrií výchozích „seed“ prostoročasů, jež často umožňují Einsteinovy-Maxwellovy rovnice výrazně zjednodušit. Stručný přehled těchto metod lze nalézt v 1. kapitole, kde je rovněž shrnut základní matematický aparát, ze kterého vycházejí veškeré výpočty i výsledky prezentované v dizertační práci. Až na dvě výjimky jsou složky všech tenzorů vždy vyjádřeny vzhledem k ortonormovaným bázím, tj. v tetrádovém formalismu. Ukazuje se, že právě v něm nejlépe vyniknou analogické vlastnosti studovaných prostoročasů a že nejlépe vyhovuje použitému způsobu získávání nových Einsteinových-Maxwellových polí.

Hypotéza Horského-Mickeviče

Ze známých generujících technik je v práci systematicky využita hypotéza o bezprostřední souvislosti mezi čtyřpotenciálem hledaného elektromagnetického pole a izometriemi výchozích „seed“ vakuových prostoročasů vyslovená v roce 1989 Horským a Mickevičem [22]. Výhodou této metody je, že klade poměrně silné požadavky na tvar čtyřpotenciálu hledaného elektromagnetického pole. Navíc získaná třída „nabitých“ řešení obsahuje oproti výchozím prostoročasům pouze jediný další parametr charakterizující intenzitu elektromagnetického pole, zatímco použití jiných generujících technik vede často k mnohoparametrickým třídám exaktních řešení, přičemž fyzikální interpretace jejich parametrů bývá obvykle dosti problematická. Tím, že v centru metody Horského a Mickeviče leží čtyřpotenciál hledaného elektromagnetického pole, máme o tomto poli od počátku maximální možné množství informací a do určité míry můžeme ovlivnit, zda výsledné pole bude

elektrického či magnetického typu, popřípadě ve speciálních případech dokonce jejich superpozicí.

Principy i aplikace metody získávání Einsteinových-Maxwellových polí založené na hypotéze Horského a Mickeviče jsou popsány ve 2. kapitole. V návaznosti na výsledky dosažené jinými autory jsou uvedeny tři konkrétní příklady prostoročasů, jež sice byly nalezeny jinou cestou, ale splňují všechny podmínky kladené generující hypotézou a nebyly v této souvislosti dosud studovány. V závěru kapitoly je nastíněno, jak je možné některých myšlenek Horského a Mickeviče použít i v jiném, do určité míry obráceném smyslu, např. pro určení čtyřpotenciálu Einsteinových-Maxwellových polí. Vyvstává zde i další otevřená otázka, zda existence odpovídající vakuové „seed" metriky nemůže být nutnou podmínkou pro to, abychom příslušné řešení Einsteinových-Maxwellových rovnic mohli pokládat za fyzikálně interpretovatelné a mohlo tak alespoň přibližně popisovat gravitační pole v okolí nabitých hmotných objektů, popř. hmotných objektů umístěných ve vnějším elektromagnetickém poli.

Přestože generující hypotéza Horského a Mickeviče v podstatě umožňuje získávat nová řešení Einsteinových-Maxwellových rovnic, je třeba zdůraznit, že hypotéza sama o sobě představuje složitý, stále otevřený geometricko-fyzikální problém. Není k dispozici žádný obecný důkaz zaručující, že aplikace hypotézy vždy povede k novému „nabitému" řešení. Kromě toho zobecněná formulace hypotézy vyslovená v roce 1994 [7] předpokládá mnohem volnější vztah mezi Killingovými vektory vakuového řešení a čtyřpotenciálem elektromagnetického pole v hledaném „nabitém" prostoročase. Na druhé straně je tato zobecněná podoba hypotézy použitelná v mnohem větším počtu případů. Zřejmě dosud úplně nerozumíme fyzikálně-geometrické podstatě problému, přestože po matematické stránce jsme schopni ho řešit, mnohdy úspěšně. Velkým problémem je rovněž skutečnost, že neexistuje algoritmus, podle něhož by bylo možné při hledání nových řešení přesně a jednoznačně postupovat. I z tohoto důvodu je třeba přikládat velký význam systematickému hledání a zkoumání co největšího počtu exaktních řešení, jejichž analýzou můžeme zpřesňovat obecné závěry, dospět k adekvátnějšímu intuitivnímu pochopení problému a proniknout hlouběji k jeho kořenům. Fundamentálním teoretickým otázkám se předložená práce záměrně vyhýbá a zaměřuje se naopak na bezprostřední praktické využití generující hypotézy.

Einsteinova-Maxwellova pole Levi-Civitova typu

Ve 3. kapitole je Horského-Mickevičova hypotéza aplikována na vakuové Levi-Civitovo řešení Einsteinových rovnic známé od roku 1919. Převážná část této kapitoly bude publikována *Czechoslovak Journal of Physics* [48], konečná verze článku byla také umístěna do elektronického archivu preprintů `http://xxx.lanl.gov/gr-qc/0003004`.

Ve 3. kapitole jsou nejprve shrnuty základní vlastnosti Levi-Civitova prostoročasu, stručně je vysvětlen význam jeho parametrů a jsou nastíněny problémy s jeho nejednoznačnou fyzikální interpretací. V části 3.2 je potom navržen algoritmus předepisující způsob, jak je možné zjednodušit aplikaci Horského-Mickevičovy hypotézy na Levi-Civitovu „seed" metriku. Ukazuje se, že každému z pěti Killingových vektorů Levi-Civitova řešení odpovídá určité řešení Einsteinových-Maxwellových rovnic. Pro všechna řešení je provedena Petrovova klasifikace a je také určen typ získaného elektromagnetického pole. Jako vedlejší produkt výpočtů pak bylo nalezeno ještě jedno „nabité" řešení. V žádném případě se však nejedná o rozpor s generující hypotézou Horského-Mickeviče, neboť nikdy nebyla považována za jediný možný způsob generování elektromagnetických řešení. Až na jedinou výjimku jsou nalezená řešení skutečně nová, interpretujeme-li je v cylindrických Weylových souřadnicích, v nichž jsou také explicitně uvedena v dizertační práci. Principiálně nelze vyloučit existenci transformací, které by nalezená řešení mohly převést na některé z již známých metrik. Problém vzájemné ekvivalence Einsteinových i Einsteinových-Maxwellových polí zdaleka není triviální a jeho řešení vyžaduje adekvátní softwarové prostředky, jež při kompletování dizertační práce nebyly k dispozici.

Pro každé z nalezených řešení je navržena jeho fyzikální interpretace. Vychází ze studia radiálních geodetických čar a z numerické simulace geodetických trajektorií v rovině kolmé na osu symetrie. Pro Einsteinova-Maxwellova pole, která lze vyjádřit ve tvaru stacionární, osově symetrické Weylovy metriky, lze nalézt odpovídající newtonovský gravitační potenciál a následně porovnat pohyb částic po geodetických čarách ve vyšetřovaných prostoročasech s pohybem klasických hmotných částic v těchto newtonovských gravitačních polích. Jejich kvalitativní shoda hovoří ve prospěch interpretace získaných řešení právě na základě analytického tvaru newtonovských gravitačních potenciálů alespoň pro určité hodnoty parametrů. V důsledku toho lze říci, že nalezená exaktní řešení Einsteinových-Maxwellových rovnic mohou popisovat vnější pole nekonečného lineárního hmotného objektu (nekonečně dlouhé, tenké „tyče“ nebo „drátu“) nabitého elektrickým nábojem, vloženého do vnějšího elektrického nebo magnetického pole, popř. lineárního objektu, kterým protéká elektrický proud. Konformní struktura Einsteinových-Maxwellových polí je znázorněna pomocí Penroseových diagramů, na nichž je vyznačena poloha prostoročasových singularit. Je třeba přiznat, že přítomnost singularit byla zjišťována pouze na základě divergence Kretschmannova skalárního invariantu, což je podmínka postačující, avšak nikoli nutná.

Obecnější Einsteinova-Maxwellova pole Weylova typu

Ve 4. kapitole se hypotéza Horského-Mickeviče aplikuje na některé další vakuové Weylovy metriky podle algoritmického schématu navrženého v kapitole 3. Nejprve jsou některé speciální případy Einsteinových-Maxwellových polí Levi-Civitova typu převedeny souřadnicovými transformacemi odvozenými v [4] na řešení Taubova, popř. Robinson-Trautmanova typu, jež zřejmě nelze považovat za nová, nabízejí však alternativní pohled na některé závěry přecházející kapitoly. Poté jsou nalezena řešení Einsteinových-Maxwellových rovnic, jejichž limitním případem jsou gravitační pole nekonečné nehomogenní roviny a „napůl nekonečného“ lineárního zdroje (tenké „tyče“ nebo tenkého „drátu“ rozložených od nějakého bodu do nekonečna); oba typy řešení jsou ekvivalentní, od jednoho ke druhému lze přejít souřadnicovou transformací nalezenou Bonnorem [6]. V závěru kapitoly je potom Horského-Mickevičova generující hypotéza aplikována na γ-metriku, vakuové řešení Einsteinových rovnic popisující gravitační pole tenké konečné tyče (ve Weylových souřadnicích) nebo sféroidu (v Erez-Rozenových souřadnicích) [21]. Získaná nabitá řešení lze potom pro určité hodnoty parametrů interpretovat jako vnější pole nabité tyče konečné délky respektive jako pole tyče vložené do podélného magnetického pole. Pro obě třídy řešení jsou nalezeny odpovídající newtonovské potenciály, jejichž tvar je ilustrován na několika obrázcích, a je vyšetřována existence i poloha prostoročasových singularit. S využitím výsledků [21] je ukázáno, že některé z prostoročasů studovaných ve 3. kapitole jsou limitním případem Einsteinových-Maxwellových polí získaných z γ-metriky. Zajímavým speciálním případem jsou pak dvě „nabitá“ řešení, jež při nulovém elektromagnetickém poli přecházejí ve známé Schwarzschildovo řešení. Vzájemné vztahy vakuových „seed“ prostoročasů jsou znázorněny limitním diagramem. Ukazuje se, že výše uvedená fyzikální interpretace získaných prostoročasů nejlépe odráží relace metrik v limitním diagramu a poskytuje tak solidní základ pro ucelenější pohled na vygenerovaná řešení Einsteinových-Maxwellových rovnic.

Shrnutí prezentovaných výsledků

V předložené práci je na praktickém použití ukázáno, že metoda generování Einsteinových-Maxwellových polí založená na generující hypotéze Horského-Mickeviče je skutečně užitečným a poměrně efektivním nástrojem pro hledání nových exaktních řešení. Výsledky a závěry lze shrnout do několika bodů.

- V práci je generující hypotéza Horského-Mickeviče ve zobecněné formulaci podle [7] systematicky aplikována na celou třídu vakuových „seed“ prostoročasů. Ve všech případech každému Killingovu vektoru odpovídá exaktní řešení Einsteinových-Maxwellových rovnic. Přestože ještě zcela nerozumíme geometrické a fyzikální podstatě generující hypotézy, počet nalezených řešení je nadějným příslibem pro další použití hypotézy a jejích základních myšlenek.

- Algoritmický postup pro generování „nabitých“ řešení formulovaný v souvislosti s Levi-Civitovou „seed“ metrikou se ukázal být použitelný i v obecnějších případech, např. při aplikaci generující hypotézy na vakuovou γ-metriku. Lze proto říci, že některé kroky tohoto postupu mohou být inspirací při hledání ještě složitějších řešení a že postup samotný představuje zajímavý příspěvek k problematice generující hypotézy v širších souvislostech.

- Bylo získáno několik tříd exaktních řešení Einsteinových-Maxwellových rovnic, z nichž většina představuje statické, osově (popř. i válcově) symetrické prostoročasy, které je možno zapsat ve Weylově tvaru. Díky tomu lze snadno nalézt odpovídající newtonovské gravitační potenciály, jež poskytují kvalitativně dobrou intuitivní představu o pohybu volných částic ve studovaných prostoročasech. Bylo ukázáno, že některá Einsteinova-Maxwellova pole lze považovat za limity obecnějších polí, některá pak lze souřadnicovými transformacemi převést na jiný tvar, který nabízí jiný kontext a jinou fyzikální interpretaci. Většina nalezených řešení je skutečně nová, pokud je interpretujeme v cylindrických válcových souřadnicích. Přesný a jednoznačný důkaz toho, že nalezená řešení nemohou být převedena na některé již známé metriky, však podán není, neboť je velmi komplikovaný, zdlouhavý a vyžaduje odpovídající programové vybavení.

- Pro všechna nalezená řešení jsou určeny složky nejdůležitějších tenzorů, typ elektromagnetického pole (elektrické nebo magnetické) a provedena Petrovova klasifikace. Dále jsou vyšetřovány množiny bodů, v nichž diverguje Kretschmannův skalár; jedná se o postačující podmínku pro existenci singularit.

- Pro nalezená řešení je vždy navržena jeho fyzikální interpretace. Pro Einsteinova-Maxwellova pole získaná pomocí generující hypotézy je příznačné, že jejich interpretaci předurčuje fyzikální interpretace výchozích „seed“ prostoročasů. Závěry prezentované v dizertační práci vycházejí především z [6], i když celá řada případů připouští i alternativní pohled. Zdá se však, že interpretace získaných řešení jako gravitačních polí v okolí nabitých konečných i nekonečných lineárních zdrojů, popř. v okolí lineárních zdrojů vložených do vnějšího magnetického pole nejen nejlépe vyhovuje principu korespondence mezi obecnou teorií relativity a Newtonovou teorií gravitace, ale velmi dobře odpovídá intuitivní představě o vzájemných limitních vztazích mezi lineárními zdroji, popř. o jejich superpozici. Kvalitativní shoda mezi tvarem geodetických čar a trajektoriemi volných částic v odpovídajících Newtonových potenciálech rovněž hovoří ve prospěch tohoto závěru. Je však nutné přiznat, že výše uvedená interpretace není přijatelná pro všechny hodnoty parametrů metrického tenzoru.

Řada problémů diskutovaných v práci přirozeně vede k novým otázkám a otevírá problémy nové, jež se bezprostředně nabízejí jako možnosti dalšího výzkumu. Kromě principiálních teoretických otázek o podstatě a případných mezích použitelnosti generující hypotézy mezi tyto problémy patří například:

- Je nutné jednoznačně dokázat, zda jsou získaná řešení skutečně nová či nikoli.
- Je třeba dokončit Petrovovu klasifikaci všech řešení, konkrétně zjistit, zda řešení získaná „nabitím“ γ-metriky a náležející obecně k Petrovovu typu I se v některých speciálních případech neredukují na algebraicky speciální Petrovovy třídy.
- Bylo by žádoucí klasifikovat nalezená řešení podle grup izometrií.
- Bylo by zajímavé detailněji studovat singularity těch řešení elektrického typu, jež mají proti výchozím „seed“ vakuovým metrikám jednu singularitu navíc .
- Pokud vyjdeme z fyzikální interpretace vakuových metrik preferované v [6], tzn. budeme-li pro určité hodnoty parametrů nalezená řešení považovat za pole nabitých lineárních zdrojů, popř. za pole lineárních zdrojů umístěných ve vnějším elektrickém či magnetickém poli, potom by bylo zajímavé spočítat integrální charakteristiky těchto zdrojů (celkový elektrický náboj apod.) v těch případech, kde je to možné a kde mají tyto charakteristiky fyzikální smysl.
- Bylo by velmi zajímavé, mimo jiné i z hlediska obecných problémů spojených s generující hypotézou Horského-Mickeviče, aplikovat generující hypotézu na některé vakuové prostoročasy s nenulovou kosmologickou konstantou. V souvislosti s typy řešení, která jsou studována v dizertační práci, se k tomuto účelu jako nejvhodnější „seed“ metrika jeví Levi-Civitovo řešení s nenulovou kosmologickou konstantou popsané v [17]. Pokud by bylo možné najít příslušné řešení Einsteinových-Maxwellových rovnic, jednalo by se o důležitý krok, neboť na vakuová řešení s nenulovou kosmologickou konstantou nebyla generující hypotéza dosud aplikována.

Appendix B

Curriculum vitae

Personal data

Name: Lukáš Richterek *Date of birth:* 11th March 1969
Place of birth: Rychnov nad Kněžnou, Czech Republic *Nationality:* Czech
Family state: married with Šárka Richterková (Dostálová) since 19th July 1997
Children: daughter Pavla, born on 9th April 1998
Private address: U cukrovaru 20, Olomouc-Holice, CZ-783 71
Private phone: +420-68-5314562

Workplace

Department of Theoretical Physics, Faculty of Natural Sciences, Palacký University
Svobody 26, Olomouc, CZ-771 46
Phone: +420-68-5222451, ext. 372 *Fax*: +420-68-5225737 (dean's office)
E-mail: `richter@risc.upol.cz`
Internet: `http://aix.upol.cz/~richter/index.html`

Education

Undergraduate
Faculty of Nat. Sciences, Palacký University Olomouc
Subject: mathematics & physics (pedagogical branch) 1987-1992
External postgraduate
Faculty of Nat. Sciences, Masaryk University Brno
Subject: general problems of physics .... started in 1994

Employment summary

Military service .... April–December 1993
Technical worker (Dept. Theor. Phys., Palacký University) 1992–1995
Assistant lecturer (Dept. Theor. Phys., Palacký University) ...... since September 1995

Pedagogical activity

Lecture courses: Introduction to general relativity, Introduction to relativistic cosmology, Special relativity, Statistical physics and thermodynamics.
Seminars: Theoretical mechanics, Special relativity, Quantum mechanics I and II, Statistical physics and thermodynamics.

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Active participation in scientific seminars

12. 5. 1995	seminar of the Relativistic Group, Department of Theoretical Physics and Astrophysics, Faculty of Science, MU Brno: "*Pencil of Light in General Relativity.*"
3. 5. 1996	seminar of the Relativistic Group, Department of Theoretical Physics and Astrophysics, Faculty of Science, MU Brno: "*Exact solutions of the Einstein-Maxwell equations with special symmetries.*"
18. 4. 1997	seminar of the Relativistic Group, Department of Theoretical Physics and Astrophysics, Faculty of Science, MU Brno: "*Some comments on the generating conjecture.*"
13. 3. 1998	seminar of the Relativistic Group, Department of Theoretical Physics and Astrophysics, Faculty of Science, MU Brno: "*Solutions of the Einstein-Maxwell equations fulfilling H-M conjecture.*"
23. 10. 1998	seminar of the Department of Theoretical Physics, Faculty of Natural Sciences, Palacký University Olomouc: "*About possible connection between the electromagnetic field and space-time symmetries.*"
9. 12. 1999	seminar of the Relativistic Group, Department of Theoretical Physics and Astrophysics, Faculty of Science, MU Brno: "*On some axisymmetric Einstein-Maxwell fields.*"
16. 12. 1999	RAGtime N°1 (a workshop an accretion processes in the field of black holes), Institute of Physics, Silesian University Opava: "*On some axisymmetric Einstein-Maxwell fields.*"
14. 3. 2000	seminar of the Relativistic Group, Department of Theoretical Physics, Faculty of Mathematics and Physics, Charles University Prague: "*Einstein-Maxwell fields of Levi-Civita's type.*"

Other proffession-related activities

Member of the Union of Czech Mathematicians and Physicists .... since 1995
Member of the American Association of Physics Teachers .... since 1996
Leader of "Olomouc correspondence seminar in physics" . September 1994 – May 1998
Chair of a regional committee of the Physics Olympiad .... since September 1998

Appendix C

List of publications

Publications related to the topic of the thesis

Richterek, L., Novotný, J., and Horský, J., 2000, "New Einstein-Maxwell fields of Levi-Civita's type," *Czech J. Phys.* (accepted for publication); also `gr-qc/0003004`.

Publications not related to the topic of the thesis

Majerník, V., and Richterek, L., 1997, "Entropic uncertainty relations for the infinite well," *J. Phys. A: Math. Gen.* **30**, L49–L54.

Majerník, V., Richterek, L., 1997, "Entropic uncertainty relations," *Eur. J. Phys.* **18**, 79–89.

Mechlová, E., Košťál, K., et al., 1999, *Výkladový slovník fyziky pro základní vysokoškolský kurz* [Physics dictionary for a university course], Prometheus, Praha, 191–200 (in Czech).

Richterek, L., and Majerník, V., 1999, "Field energy in classical gravity," *Acta UPO - Fac. Rer. Nat., Physica* **38**, 47–58.

Popularization articles (in Czech)

Richterek, L., 1996, "Hubbleův kosmický dalekohled zahájil útok na Hubbleovu konstantu" [HST has started an attack on the Hubble constant], *Říše hvězd* [Realm of stars] **76**, No. 11-12, 208–209.

Majerník, V., Richterek, L., and Vlček, V., 1999, "Entropické relace neurčitosti" [Entropic uncertainty relations], *Pokroky matematiky, fyziky a astronomie* [Progress in mathematics, physics and astronomy] **44**, No. 4, 315–334.

Majerník, V., and Richterek, L., 1999, "Lorentzovy transformace trochu netradičně" [Lorentz transformations in a little non-traditional way], *Čs. čas. fyz.* [Czechoslovak journal for Physics] **49**, No. 5, 305–318.

Richterek, L., 1995-2000, "Zajímavé úlohy z fyziky" [Interesting problems in physics], regular section of *Matematika – fyzika – informatika* [Mathematics – Physics – Informatics] **5–9**, Prometheus, Praha.

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