OPTIMAL PARTIAL ESTIMATION OF QUANTUM STATES

Theory: **Ladislav Mišta**, **Radim Filip**, **Jaromír Fiurášek**

Department of Optics, Palacký University, Czech Republic.

Experiment: **Fabio Sciarrino**, **Marco Ricci**, **Francesco De Martini**

Department of Physics, University of Roma "La Sapienza", Italy.

Supported by project
"Measurement and Information in Optics" (MSM 6198959213)

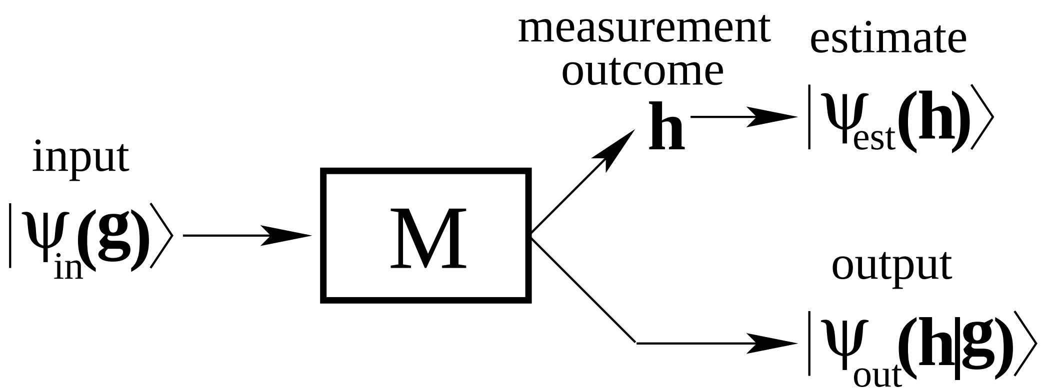
Optimal fidelity trade-off

Measurement of a quantum state disturbs the state.
The more information is gained, the bigger is the disturbance.

$\downarrow$

Permitted trade-off between the information gain and state disturbance.

What measurements introduce the least possible disturbance?



State disturbance-mean output fidelity F.
Information gain-mean estimation fidelity G.

Optimal trade–off between F and G?
(K. Banaszek, PRL 86, 1366 (2001).)

Other measures: s.d.–Bures-Uhlmann fidelity, i.g.–Shannon entropy.
(K. Banaszek, arXiv:quant-ph/0006062.)

Assumptions:

Input states: $|\psi(g)\rangle = U(g)|\psi(0)\rangle$, $g \in$ group $\mathcal{G}$, $U(g)$ is UI representation of $\mathcal{G}$.

A-priori distribution of input states: normalized invariant measure dg on $\mathcal{G}$.

Each measurement outcome can be labelled by a group element h.

Method:

Each outcome $h \in \mathcal{G} \rightarrow$ trace-decreasing CP map $\chi(h)$ on $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$ (dim$\mathcal{H} = d$).

Output state: $\rho(h|g) = \text{Tr}_{\text{in}}[\chi(h)\psi(g)^T \otimes 1_{\text{out}}]$, $\text{Tr}_{\text{out}}[\rho(h|g)] = P(h|g)$.

(Trace-preservation constraint $\int_g \text{Tr}_{\text{out}}[\chi(g)]dg = 1_{\text{in}}$).

Mean output fidelity $F = \int_g \int_h \langle\psi(g)|\rho(h|g)|\psi(g)\rangle dh dg$.

Estimated state (if the otcome was h): $|\psi(h)\rangle$.

Mean estimation fidelity $G = \int_g \int_h P(h|g)|\langle\psi(g)|\psi(h)\rangle|^2 dh dg$.

Covariance (w.l.o.g.): $\chi_{\text{cov}}(g) = [U^*_{\text{in}}(g) \otimes U_{\text{out}}(g)]\chi_0[U^T_{\text{in}}(g) \otimes U^\dagger_{\text{out}}(g)]$.

$$\downarrow$$

$$F = \text{Tr}[\chi_0 R_F], \qquad G = \text{Tr}[\chi_0 R_G],$$

$$R_F = \int_g \psi(g)^T \otimes \psi(g) dg, \qquad R_G = \text{Tr}_{\text{out}}[R_F 1_{\text{in}} \otimes \psi(0)] \otimes 1_{\text{out}}.$$

Maximization of G for a fixed $F \Leftrightarrow$ maximization of $\mathcal{F} = pF + (1-p)G$, $p \in [0,1]$.

$\mathcal{F} \leq d r_{p,\mathrm{max}}$, where $r_{p,\mathrm{max}}$ is maximum eigenvalue of $R_p = pR_F + (1-p)R_G$.

Optimal $\chi_0 = |\chi_0\rangle\langle\chi_0|$, where $|\chi_0\rangle$ is an eigenvector corresponding to $r_{p,\mathrm{max}}$.

Partial estimation of completely unknown state

Input states: All pure states $|\psi\rangle$ of a d-level system, i.e. $\mathcal{G} = SU(d)$.

Reference state: $|\psi(0)\rangle = |+\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |j\rangle$, $\{|j\rangle\}_{j=1}^{d}$ ON basis.

$$\downarrow$$

$$R_F = \frac{1}{d(d+1)} \left(1_{\mathrm{in}} \otimes 1_{\mathrm{out}} + d|\Phi^+\rangle\langle\Phi^+| \right), \; |\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |jj\rangle,$$

$$R_G = \frac{1}{d(d+1)} (1_{\mathrm{in}} \otimes 1_{\mathrm{out}} + |+\rangle_{\mathrm{in}}\langle+| \otimes 1_{\mathrm{out}}).$$

Optimal map: $|\chi_0\rangle = \sqrt{d}\left(\alpha|+\rangle|+\rangle + \beta|\Phi^+\rangle\right)$, $\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sqrt{d}} = 1$, $\alpha, \beta \geq 0$.

$$\boxed{\sqrt{F - \frac{1}{d+1}} = \sqrt{G - \frac{1}{d+1}} + \sqrt{(d-1)\left(\frac{2}{d+1} - G\right)}}$$

(K. Banaszek, PRL 86, 1366 (2001).)

Partial estimation with prior partial information

Input states: $|\psi(\phi)\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^{d} e^{i\phi_j}|j\rangle$, $\phi_j \in [0, 2\pi)$, $j = 1, \ldots, d$, i.e. all states produced by d independent phase shifts of the state $|\psi(0)\rangle = |+\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^{d}|j\rangle$, i.e. $\mathcal{G}$ is a tensor product of d Abelian groups U(1).

A-priori distribution of input states: $dg = (2\pi)^{-d} d\phi_1 \ldots d\phi_d$.

$$\downarrow$$

$$R_F = \frac{1}{d^2}\left(1_{\text{in}} \otimes 1_{\text{out}} + d|\Phi^+\rangle\langle\Phi^+| - \sum_{j=1}^{d}|jj\rangle\langle jj|\right),$$
$$R_G = \frac{1}{d^2}\left[\left(1 - \frac{1}{d}\right) 1_{\text{in}} \otimes 1_{\text{out}} + |+\rangle_{\text{in}}\langle +| \otimes 1_{\text{out}}\right].$$

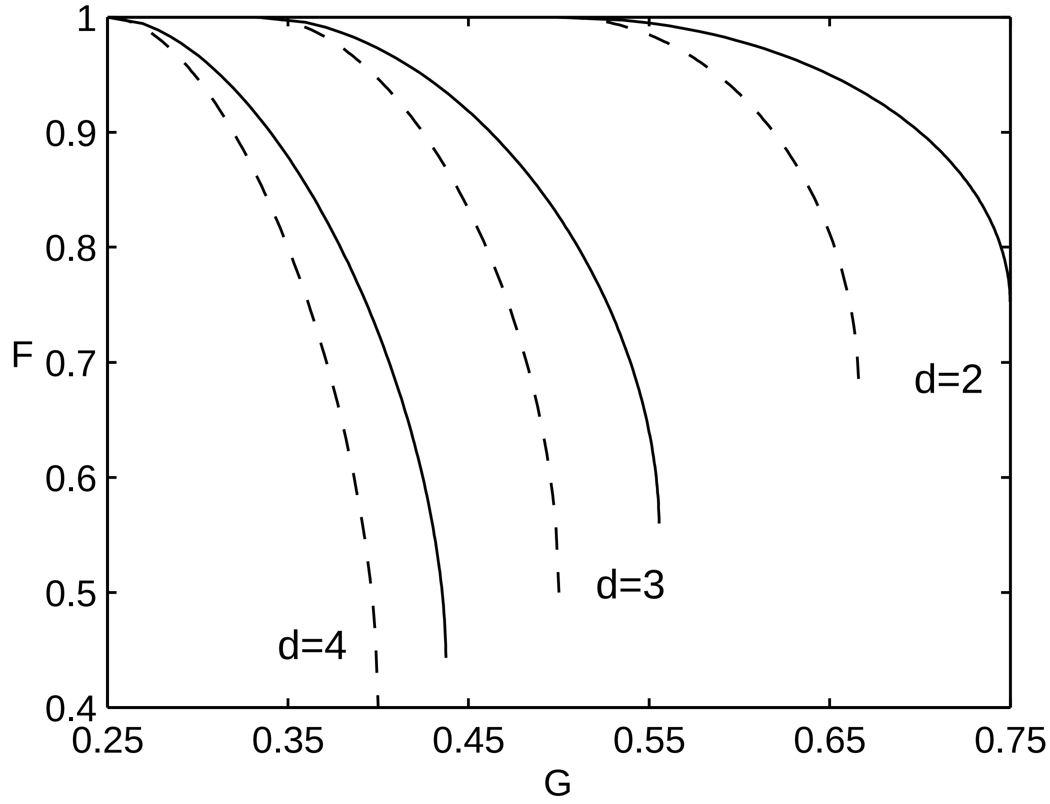
Optimal map: $|\chi_0\rangle = \sqrt{d}\left(\alpha|+\rangle|+\rangle + \beta|\Phi^+\rangle\right)$, $\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sqrt{d}} = 1$, $\alpha, \beta \geq 0$.

$$\downarrow$$

Optimal fidelity trade-off

$$\boxed{\sqrt{d^2 G - d + 1} = \sqrt{d(1 - F)} + \sqrt{\frac{dF - 1}{d - 1}}}$$

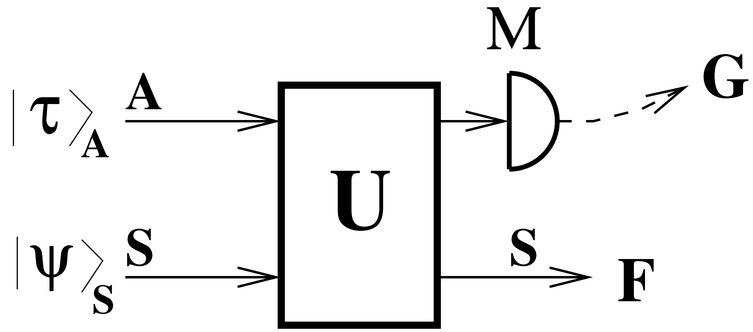
(L. Mišta, J. Fiurášek, and R. Filip, PRA 72, 012311 (2005).)



Optimal trade-off between F and G for a partially known state (solid curve) and a completely unknown state (dashed curve).

Implementation of optimal partial measurement

- Completely unknown input state:
  By teleportation with nonmaximally entangled states.
  (K. Banaszek, PRL 86, 1366 (2001).)
  By QND-type measurement with a single properly prepared ancilla.
  (L. Mišta and R. Filip, PRA 72, 034307 (2005).)



Input state: $|\psi\rangle_S = \sum_{j=1}^{d} c_j |j\rangle_S$, $\{|j\rangle_S\}_{j=1}^{d}$ ON basis.

Unitary U is such that: 1. $|\psi\rangle_S|\mu\rangle_A \xrightarrow{U} \sum_{j=1}^{d} c_j |j\rangle_S |\mu_j\rangle_A$, $\{|\mu_j\rangle_A\}_{j=1}^{d}$ ON basis.

2. $|\psi\rangle_S|\kappa\rangle_A \xrightarrow{U} |\psi\rangle_S \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |\mu_j\rangle_A$.

State of ancilla: $|\tau\rangle_A = \alpha|\mu\rangle_A + \beta|\kappa\rangle_A$, $\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sqrt{d}} = 1, \alpha, \beta \geq 0$.

Measurement M: Measurement of A in the basis $\{|\mu_j\rangle_A\}_{j=1}^{d}$.

$$\downarrow$$

Measurement operators: $A_r = \alpha|r\rangle_S\langle r| + \frac{\beta}{\sqrt{d}}1_S,\ r = 1,\ldots,d.$

Partial nondemolition measurement of $\{|j\rangle_S\}_{j=1}^{d}$.

Reconstruction strategy: If $|\mu_r\rangle_A$ is measured then our estimate is $|r\rangle$.

$$\downarrow$$

$F = 1 - \frac{d-1}{d+1}\alpha^2$ and $G = \frac{2}{d+1} - \frac{d-1}{d(d+1)}\beta^2$ saturate Banaszek's trade-off.

U can be realized by d-dimenzional generalization of the CNOT gate.

$U_{\text{CNOT},d}|i\rangle_S|j\rangle_A = |i\rangle_S|i\oplus j\rangle_A,\ i,j = 0,\ldots,d-1;\ |\mu\rangle_A = |0\rangle_A;\ |\kappa\rangle_A = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle_A.$

- Partially known input state:
  By partial nondemolition measurement in a suitable basis.
  (L. Mišta, J. Fiurášek, and R. Filip, PRA 72, 012311 (2005).)

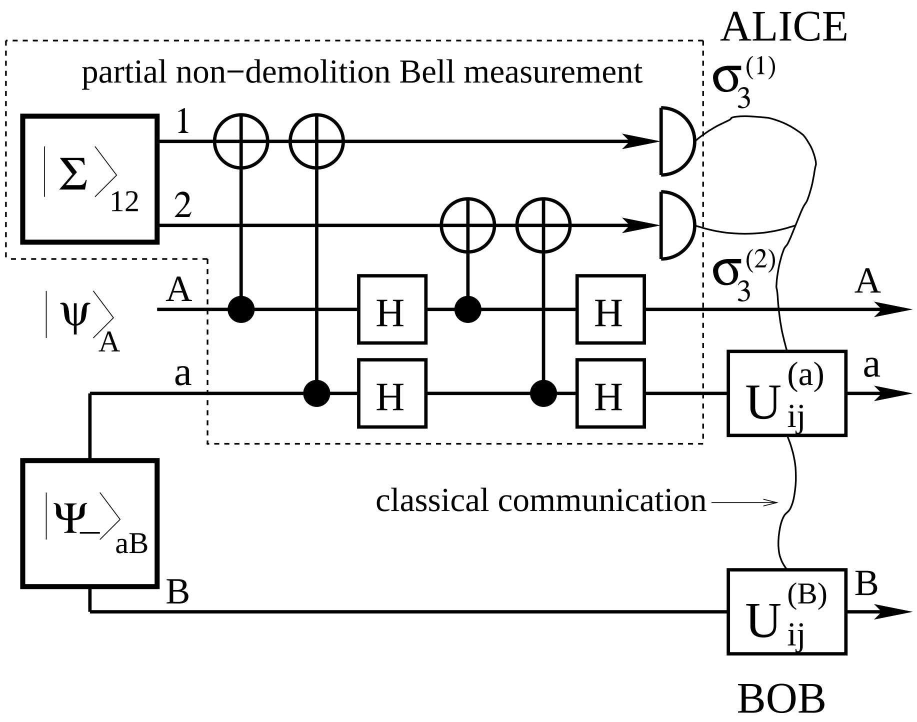
Input state: $|\psi(\phi)\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} e^{i\phi_j}|j\rangle$.
Measurement: Partial nondemolition measurement of $\{|\tilde{j}\rangle_S \equiv F|j\rangle_S\}_{i=1}^{d}$.
($F = \frac{1}{\sqrt{d}} \sum_{j,k=1}^{d} \exp(i2\pi/d)^{jk}|j\rangle\langle k|$ Finite-dimensional Fourier transform.)
Reconstruction strategy: If $|\mu_r\rangle_A$ is measured then our estimate is $|\tilde{r}\rangle$.

$$\downarrow$$

$$F = 1 - \left(\frac{d-1}{d}\right)^2 \alpha^2 \text{ and } G = \frac{2d-1}{d^2} - \frac{d-1}{d^2}\beta^2 \text{ saturate our trade-off.}$$

Experimental demonstration for qubits ($d = 2$) and equatorial qubits.
(F. Sciarrino, M. Ricci, F. De Martini, R. Filip, and L. Mišta, Jr., arXiv:quant-ph/0510097.)

Optimal partial estimation of two qubits



Input states: pure states of two qubits A and a.

Partial nondemolition Bell meas. = Complete nondemolition Bell meas. (G.-P. Guo et al., PLA 286, 401 (2001).)+ ancilla in the state:
$|\Sigma\rangle_{12} = \alpha|00\rangle_{12} + \beta|++\rangle_{12}$, $\alpha^2 + \beta^2 + \alpha\beta = 1, \alpha, \beta \geq 0$.

Reconstruction strategy:

If $|00\rangle_{12}$ is measured then our estimate is $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

If $|01\rangle_{12}$ is measured then our estimate is $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

If $|10\rangle_{12}$ is measured then our estimate is $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

If $|11\rangle_{12}$ is measured then our estimate is $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

$$\downarrow$$

$F = \frac{1+(\alpha+2\beta)^2}{5}$, $G = \frac{1+(\alpha+\beta/2)^2}{5}$ saturate Banaszek's trade-off for $d = 4$.

$$\downarrow$$

Partial nondemolition Bell meas. is optimal partial measurement for two qubits.

(L. Mišta and R. Filip, PRA 71, 022319 (2005).)

Conclusions

- General method for derivation of the optimal fidelity trade-off.
- Derivation of the optimal trade-off for partially known d-level system.
- Proposal of schemes for optimal partial measurement.
- Experimental realization of minimum disturbance measurement.