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Quantum Phase in Interferometry

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Quantum estimation of a phase shift is formulated within quantum mechanics and information theory. Using Bayes' estimation, the phase can be attributed even to a single interfering particle. The method is demonstrated for neutron interferometry operating near the quantum limit. [S0031-9007(96)00382-1]

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Though interferometry has a long track record as a useful technique for research, the quantum limitation of phase measurement must be acknowledged as an open problem in quantum mechanics and one which engages the attention of theoreticians and experimentalists alike [1]. To date, only a few experiments have been published concerning phase-shift measurements at low energies limited by quantum fluctuations [2–5], and the results of those published were sometimes affected by the particular statistical treatment. For example, only the root-mean-square deviation of the phase shift was evaluated, phase was implicitly assumed as a quantum variable with discrete spectrum, some registered data were neglected, or the phase-shift invariance of resulting distribution was anticipated [6]. The phase estimation even for very few detected particles is reported for the first time in this Letter. The quantum phase problem in interferometry is addressed from the viewpoints of quantum mechanics and of information theory. As a main goal, phase information ascribed to a single particle is quantified using Bayes' (B) estimation. On the other hand, the ultimate phase resolution may be achieved using the maximum likelihood (ML) estimation for a large mean number of recorded particles $N \gg 1$. Depending only on the *counted* statistics of the particles, the theory is valid for both bosons and fermions. The method is illustrated on the ideal interferometer. In comparison with the approaches already used in quantum mechanics [4] based on the correspondence principle, the proposed treatment emphasizes the information content included in phase sensitive data. The theory is used for analysis of phase estimation in neutron interferometry [3], offering an excellent possibility to operate in the quantum regime when only a few particles are detected.

The scheme considered here is the Mach-Zehnder interferometer as is generally used in neutron interferometry (Fig. 1). An unknown phase shift θ characterizes the path difference in both arms of the interferometer. Classically, the measured signal represented by the mean number of particles depends on the induced phase shift, yielding interference fringes on the outputs. This may be described

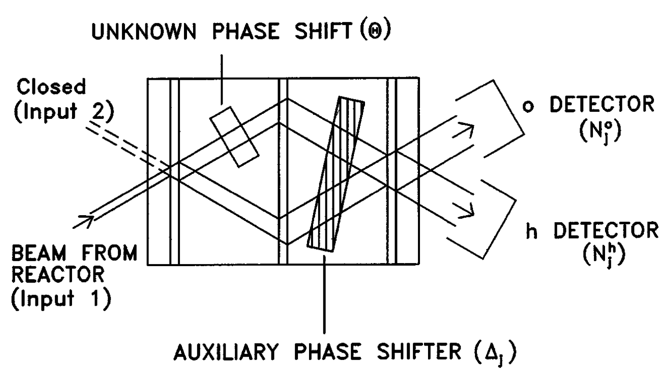

FIG. 1. Scheme of a perfect crystal neutron interferometer of the Mach-Zehnder type.

within quantum mechanics and mathematical statistics in the following way. The estimated phase shift θ, so-called *true value of the phase shift*, represents an unknown but nonrandom c-number displacement parameter. It should be determined by evaluation of measured phase sensitive data, denoted formally as a quantum variable y_k. Here the number of particles detected on the o and h output ports at m positions of an auxiliary phase shifter serve this purpose. The index k represents a detected combination of counting numbers, $k \equiv \left\{N_1^o, N_1^h, \ldots, N_m^o, N_m^h\right\}$. The phase sensitivity is manifested in the counting distribution of possible outputs as $p_k(\theta)$. The operator inducing the phase shift $\hat{N}$ is given by the difference in the number of particles in both arms of the interferometer and the phase-shift transformation of the input state $|\psi\rangle$ is $|\psi(\theta)\rangle = e^{-i\theta\hat{N}}|\psi\rangle$ [6]. Hence quantum mechanics predicts the conditional probability of output y_k when the true phase shift is θ as $p_k(\theta) = |\langle y_k|\psi(\theta)\rangle|^2$. The number of detected particles and its distribution plays a crucial role replacing the notion of classical signal. For input Poissonian statistics the output remains Poissonian with phase dependent mean numbers $\bar{N}_j(\theta)$,

$$p_{j,N_j}(\theta) = e^{-\bar{N}_j(\theta)} \frac{[\bar{N}_j(\theta)]^{N_j}}{N_j!}, \qquad (1)$$

where the index j enumerates different positions of the auxiliary phase shifter. The formulation is common for both fermions and bosons, even if formal differences appear in quantization. Fermions enter the interferometer always one by one distinguished by their quantum numbers, and all the operators and quantum states must be constructed using corresponding one-particle quantities. However, detectors register the total number of particles accumulated in a given detection time not distinguishing internal quantum numbers. On the other hand, bosons may enter as many-particle states. For example, the projection of output coherent state $|\psi(\theta)\rangle = |[\bar{N}_j(\theta)]^{1/2}\rangle_{\mathrm{coh}}$ into the Fock basis $|y_j\rangle = |N_j\rangle_{\mathrm{Fock}}$ yields the Poissonian statistics (1). In more mathematical language, interferometers are described by the SU(2) symmetry group, generators of which may be constructed using both the commuting or anticommuting annihilation operators [7]. Nevertheless, these formal aspects are beyond the scope of this contribution.

The phase sensitive data, represented formally by values $\{k_i\}$, $i = 1, \ldots, n$, are collected during n repetitions of the counting. Such measurements may serve for evaluation of the phase shift, provided that identical conditions are kept. If the ensemble of all repeated measurements is large enough, the particular variation $\{k_i\}$ appears with the probability

$$p(\{k_i\}|\theta) = p_{k_1}(\theta) \cdots p_{k_n}(\theta).$$

Denoting formally the phase estimation using detected data $\{k_i\}$ as $p(\phi|\{k_i\})$, the average conditional phase distribution of the *inferred phase shift* ϕ, when θ was valid using measurements repeated n times, reads

$$\bar{P}_n(\phi|\theta) = \sum_{\{k_i\}} p(\phi|\{k_i\}) p(\{k_i\}|\theta). \qquad (2)$$

Here the n summations independently deplete all possible values for each k_i. The statistics of the inferred variable ϕ is fully determined by the statistics of the measured events $p_k(\theta)$ and by the strategy applied to the evaluation of phase sensitive data represented by the distribution $p(\phi|\{k_i\})$. Two methods will be suggested in this Letter. (i) Estimation based on Bayes' theorem represents an interval estimation and may be regarded as a distribution of probability in the sense of degrees of belief [8]. The estimation of phase sensitive data has the form of a likelihood function

$$p_B(\phi|\{k_i\}) = \frac{1}{C_{\{k_i\}}} \prod_{i=1}^{n} p_{k_i}(\phi), \qquad (3)$$

where $C_{\{k_i\}} = \int_0^{2\pi} \prod_i^n p_{k_i}(\phi) d\phi$. Assuming the relations (2) and (3) to be valid, the explicit dependence of the estimation on the true and inferred phase shifts reads

$$\bar{P}_n(\phi|\theta) = \sum_{\{k_i\}} \frac{1}{C_{\{k_i\}}} \prod_{i=1}^{n} p_{k_i}(\theta) p_{k_i}(\phi). \qquad (4)$$

(ii) Alternatively, the maximum likelihood estimation [9] representing point estimation may be used. The detection of data $\{k_i\}$ is interpreted as the phase $\phi_{\{k_i\}}$, maximizing the probability distribution (3)

$$p_{\mathrm{ML}}(\phi|\{k_i\}) = \delta(\phi - \phi_{\{k_i\}}). \qquad (5)$$

Assuming the limit of strong field (or, equivalently, that n is sufficiently large), the ML analysis achieves the resolution predicted by the Fisher information $\Delta\phi = 1/\sqrt{nI}$, where $I = \sum_k [p_k(\theta)']^2/p_k(\theta)$, with the prime denoting the derivation $d/d\theta$. For this limit the phase distribution (2) reduces to the product over all the possible values of k [6],

$$\bar{P}_{\lim}(\phi|\theta) = \frac{1}{C_n(\theta)} \left\{\prod_k [\, p_k(\phi)]^{p_k(\theta)}\right\}^n, \qquad (6)$$

where $C_n(\theta) = \int_0^{2\pi} d\phi \{\prod_k [p_k(\phi)]^{p_k(\theta)}\}^n$. This distribution characterizes the spread of the point estimations. Nevertheless, the discrete estimator converges to the true value of the phase only for a sufficiently high number of particles registered in the interferometric experiment. If this is not true and if there are not enough detected particles, the application of point estimations is meaningless, since the convergence is no longer guaranteed.

Before approaching the experimental part, the theory will be illustrated through the simple but theoretically worthwhile example of a single mode ideal interferometer with 50/50 lossless beam splitters and closed input port 2. Since measurement in the interval $[0, 2\pi]$ will be assumed, the counting at phase shifts θ and $\theta -$

$\pi/2$, denoted here by the counting numbers N_1^o, N_1^h and N_2^o, N_2^h, will be considered. These numbers correspond to the outputs N_3, N_4, N_5, and N_6 in scheme 2 [4]. The dependence of the average number of counts on the induced phase shift determines the interference fringes as

$$\bar{N}_1^{o,h}(\theta) = \frac{\bar{N}_{\mathrm{in}}}{2}(1 \pm \cos\theta),$$

$$\bar{N}_2^{o,h}(\theta) = \frac{\bar{N}_{\mathrm{in}}}{2}(1 \pm \sin\theta),$$

$\bar{N}_{\mathrm{in}}$ being the total number of particles feeding the open interferometer input port. The limiting distribution function (6) may be reduced to a simple form

$$\bar{P}_{\mathrm{lim}}(\phi|\theta) \propto \prod_{j,b} [\bar{N}_j^b(\phi)]^{n\bar{N}_j^b(\theta)}, \tag{7}$$

with the product over $j = 1, 2$ and $b = o, h$. The predicted distribution represents an envelope of histograms of discrete phase estimations. Its form is independent of the input statistics and depends on the mean number of particles in the output only [10]. For Poissonian statistics, both inferred Bayes' and ML distributions (4) and (7) depend only on the total number of particles $N = n\bar{N}_{\mathrm{in}}$. In this case, a multiple measurement with very weak input gives the same phase prediction as the single measurement with an equivalent total particle number. Significantly, the content of phase information may always be defined for Bayes' estimation, while the point estimations become moot in the case of weak fields. On the other hand, in the asymptotic regime, the ML estimation (6) yields the resolution $1/\sqrt{N}$, whereas Bayes' estimation (4) gives the value $\sqrt{2/N}$ only. The method already used in quantum mechanics for quantum phase interpretation [4] is closely related to this treatment. Any "single" detection of $k \equiv \{N_1^o, N_1^h, N_2^o, N_2^h\}$ is semiclassically interpreted as detection of the phase ϕ_k,

$$e^{i\phi_k} = [N_1^o - N_1^h + i(N_2^o - N_2^h)] \Big/ \sqrt{(N_1^o - N_1^h)^2 + (N_2^o - N_2^h)^2}.$$

This corresponds to the ML estimation only for strong fields. Further details will be given elsewhere [10].

Under real circumstances, the experimental conditions are far away from ideal theoretical assumptions. Significantly, the statistical theory of phase estimation developed above is still valid, since the existence of interference fringes and Poissonian statistics at the output is enough to justify its usage. Besides the phase shift, the properties of the source and interferometer (mean number of particles and visibility) may also be inferred from the measured data. Below, only the single output (o beam) which yields a higher visibility than the neglected port (h beam) will be used. The standard measurement of interference fringes in neutron optics is performed in m discrete positions of an auxiliary phase shifter with controlled values of $\Delta_j; j = 1, 2, \ldots, m$ and hence $k = \{N_1, \ldots, N_m\}$. Each integer N_j characterizes the number of particles counted at the position $\theta_j = \theta + \Delta_j$ fluctuating with Poissonian distribution. The average number of particles is

$$\bar{N}_j(\theta) = \bar{N}[1 + V\cos(\theta + \Delta_j)]. \tag{8}$$

Here $\bar{N}$, which is the mean intensity of the output, and V, the visibility of the interference fringes, are assumed to be *a priori unknown* parameters, which may be estimated, together with the phase shift, on the basis of the performed counting. The estimation of phase shift ϕ related to the particular measurement $\{N_j\}$ is then given as

$$p_B(\phi|\{N_j\}) \propto \int d\bar{N} \int dV \prod_{j=1}^{m} p_{j,N_j}(\phi; \bar{N}, V). \tag{9}$$

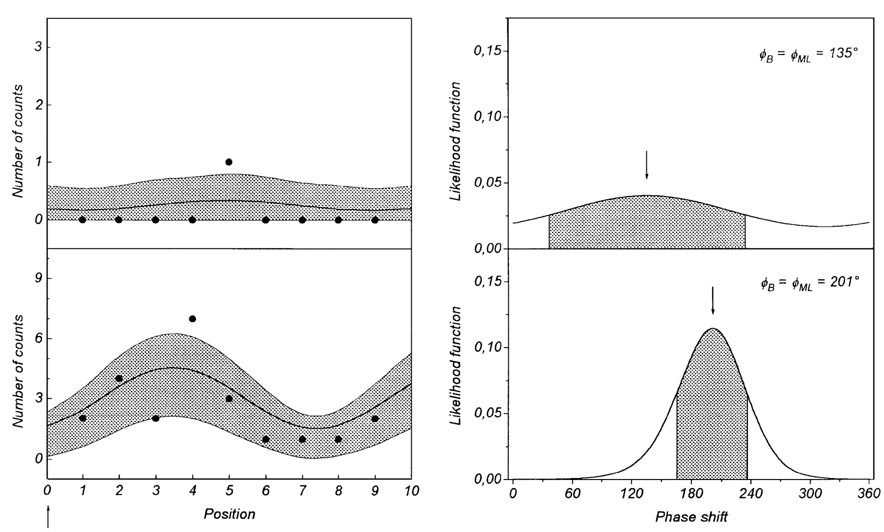

FIG. 2. Single measurement and its phase evaluation. The left panels show two single measurements selected from the ensembles with mean particle numbers $\bar{N} = 0.25$ (upper panel) and $\bar{N} = 3.0$ (lower panel) counted at nine positions of the phase shifter. The right panels show the results of Bayes' estimation [Eq. (3)], including the 68.3% confidence intervals (gray regions), and ML estimation [Eq. (5)] denoted with arrows.

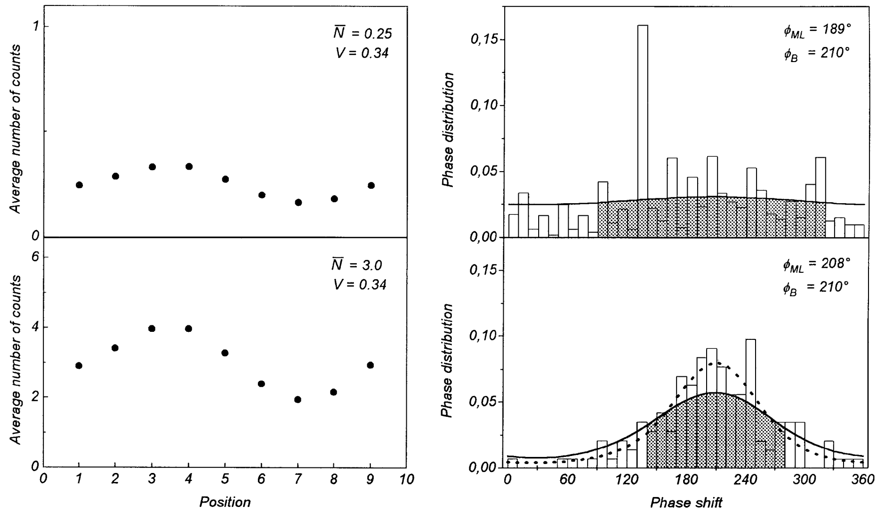

FIG. 3. Average of many repeated measurements. The left panels display the average of many single measurements with mean particle numbers $\bar{N} = 0.25$ and $\bar{N} = 3.0$ repeated 1721 times and 143 times, respectively. The solid lines on the right panels show the average of all single ($n = 1$) distribution functions [Eq. (4)] while the histograms show the average of ML estimations. The 68.3% confidence interval is denoted by gray regions. The dotted line on the right bottom panel predicts the theoretical envelope of the histograms [Eq. (6)]. Note the difference between Bayes' and ML estimation for small particle numbers where the ML estimation fails. The true value θ is 210°.

All experiments were performed at the 250 kW TRIGA reactor in Vienna. The experimental setup (Fig. 1) consists of a perfect crystal interferometer with one input beam [11], a perfect crystal phase shifter, and two neutron detectors for the o and h beams. The neutron interferometer is not lossless, but losses at the mirrors do not influence the statistics of the beams. The average counting rate in the o beam was 0.7 neutron/sec and in the h beam was 1.8 neutron/sec. The detection efficiency is nearly 1. All counted events are independent, and the stationary thermal source leads to Poissonian fluctuations of the counting numbers in both beams [3]. By rotating the phase shifter, the phase difference may be adjusted arbitrarily within the interval $[0, 2\pi]$. In the experimental setup the true value of the phase shift θ was 210°. The measurement was performed in nine positions Δ_j, as shown in the left panels of Figs. 2 and 3. This "single measurement" ($n = 1$) is repeated many times under the same conditions. The measurement with $\bar{N} = 0.25$ particles was done 1721 times, whereas the detection with $\bar{N} = 3.0$ was repeated 143 times. Each measurement serves as an evaluation of the overall phase shift θ. In addition, it also determines the mean particle number $\bar{N}$ and the visibility V. The content of phase information corresponding to the given quantum state is determined by averaging over all inferred phase distributions. The increasing number of counts obviously leads to narrower and more peaked phase distributions. The phase information may also be evaluated using Eq. (9) for very weak fields (see Figs. 2 and 3).

The statistical analysis of the phase shift in interferometers was provided free of any assumptions which cannot be verified experimentally. The distribution of the inferred phase shift was fully determined by the measured dependence of counted distribution on the induced phase shift only. The phase information may also be evaluated for very weak input fields, when point estimations fail. Particularly, the phase distribution may be attributed to a single particle interfering with itself.

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