How Quantum Correlations Enhance Prediction of Complementary Measurements

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Abstract. If there are correlations between two qubits then the results of the measurement on one of them can help to predict measurement results on the other one. It is an interesting question what can be predicted about the results of two complementary projective measurements on the first qubit. To quantify these predictions the complementary *knowledge excesses* are used. A non-trivial constraint restricting them is derived. For any mixed state and for arbitrary measurements the knowledge excesses are bounded by a factor depending only on the maximal violation of Bell's inequalities. This result is experimentally verified on two-photon Werner states prepared by means of spontaneous parametric down-conversion.

INTRODUCTION

Immediately after the discovery of quantum mechanics, it was realized that quantum correlations between two particles exhibit interesting counterintuitive features [1]. Assuming a pair of maximally entangled qubits S and M, the results of complementary measurements on qubit S can, in principle, be perfectly predicted from two appropriate measurements on qubit M. Later, it was shown that quantum mechanics predicts different values of certain correlations of measurement results than local realistic theories. Inequalities, which have to be satisfied within the local realism, were derived by Bell [2]. The predictions of quantum mechanics were already satisfactorily experimentally confirmed using pairs of photons entangled in polarizations [3, 4, 5]. In this Paper, we analyze in detail how the correlations between the qubits prepared in a general mixed state enhance our ability to predict the results of complementary projective measurements on one qubit when we know the measurement results on the other one. This enhancement can be described by the quantity that we will call complementary knowledge excess. We derive a non-trivial bound on the knowledge excesses which is determined only by the maximal violation of Bell inequalities [6]. An experimental test of this restriction on complementary knowledge excesses was performed using mixed two-photon Werner states prepared by means of spontaneous parametric down-conversion.

THEORY

We assume a general mixed state ρ_{SM} of a "signal" qubit S and a "meter" qubit M. Performing two (ideal) projective measurements Π_M, Π'_M on qubit M, the prediction of the results of mutually complementary measurements Π_S, Π'_S on qubit S can be improved. Complementarity of measurements on a qubit means that $\mathrm{Tr}\,\Pi_{Si}\Pi'_{Sj} = 1/2$ for any $i, j = 0, 1$ (Π_{Si}, Π'_{Si} are corresponding projectors). Assuming $\Pi_{S0} = |\Psi\rangle_S\langle\Psi|, \Pi_{S1} = |\Psi^{\perp}\rangle_S\langle\Psi^{\perp}|$, we can expand the state ρ_{SM} in the form $\rho_{SM} = w|\Psi\rangle_S\langle\Psi| \otimes \rho_M + w^{\perp}|\Psi^{\perp}\rangle_S\langle\Psi^{\perp}| \otimes \rho_M^{\perp} + \sqrt{ww^{\perp}}\left(|\Psi\rangle_S\langle\Psi^{\perp}| \otimes \chi_M + \text{h.c.}\right)$, where $0 \leq w, w^{\perp} \leq 1$, $w + w^{\perp} = 1$ and the meter operators $\rho_M, \rho_M^{\perp}, \chi_M$ depend on the choice of the measurement Π_S. In order to predict the result of the measurement Π_S one needs to discriminate between the mixed states ρ_M and $\rho_M^{\perp}$ by a projective two-component measurement $\Pi_M \equiv \{\Pi_{M0}, \Pi_{M1}\}$ ($\Pi_{M0} + \Pi_{M1} = 1$, $\Pi_{M0}\Pi_{M1} = 0$) on the qubit M. Using maximum likelihood estimation strategy, we can guess for each detection event the most likely result of the measurement Π_S. Our knowledge can be quantified as the fractional excess of the right guesses over wrong guesses in many such

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experiments repeated under identical conditions [7]. Using our expansion of ρ_{SM}, the total knowledge is $\mathbf{K}(\Pi_M \rightarrow \Pi_S) = \sum_i |\mathrm{Tr}_M \Pi_{Mi}(w\rho_M - w^{\perp}\rho_M^{\perp})|$, whereas without the measurement Π_M, the knowledge is $\mathbf{P}(\Pi_S) = |w - w^{\perp}|$. The largest value of knowledge over all Π_M was introduced as distinguishability $\mathbf{D}(\Pi_S) = \mathrm{Tr}_M|(w\rho_M - w^{\perp}\rho_M^{\perp})|$. Further, we define a knowledge excess

$$\Delta\mathbf{K}(\Pi_M \rightarrow \Pi_S) = \mathbf{K}(\Pi_M \rightarrow \Pi_S) - \mathbf{P}(\Pi_S), \tag{1}$$

where $0 \leq \Delta\mathbf{K}(\Pi_M \rightarrow \Pi_S) \leq 1$. It quantifies only that amount of the knowledge which exceeds the a-priori knowledge $\mathbf{P}(\Pi_S)$. The largest $\Delta\mathbf{K}(\Pi_M \rightarrow \Pi_S)$ over all Π_M can be considered as a distinguishability excess $\Delta\mathbf{D}(\Pi_S)$. Thus $0 \leq \Delta\mathbf{K}(\Pi_M \rightarrow \Pi_S) \leq \Delta\mathbf{D}(\Pi_S)$. Analogical quantities $\Delta\mathbf{K}(\Pi'_M \rightarrow \Pi'_S)$ and $\Delta\mathbf{D}(\Pi'_S)$ can be defined for the complementary measurement Π'_S.

With the help of unitary tranformations we can finally prove that

$$\Delta\mathbf{K}^2(\Pi_M \rightarrow \Pi_S) + \Delta\mathbf{K}^2(\Pi'_M \rightarrow \Pi'_S) \leq \left(\frac{B_{\max}}{2}\right)^2. \tag{2}$$

For details of the proof see Ref. [9].

In the inequality (2) the maximal Bell factor represents a non-trivial bound on the sum of the squares of knowledge excesses which can be extracted from a pair of measurements on the "meter" qubit. Assuming $\Pi_M = \Pi'_M$ we can also derive an inequality analogous to that given in Ref. [7]: $\Delta\mathbf{K}^2(\Pi_M \rightarrow \Pi_S) + \Delta\mathbf{K}^2(\Pi_M \rightarrow \Pi'_S) \leq 1$. Our analysis shows that for $\Pi_M \neq \Pi'_M$ the unit value on the right-hand side may be overstepped. Note also that $(B_{\max}/2)^2 > 1$ only if the state violates Bell inequalities.

A natural question is how inequality (2) can be saturated. For the class of states with vanishing a-priori knowledges for any measurements Π_S, Π'_S it can be saturated just by the appropriate choice of measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$. In fact, it corresponds to the transformation of the given state to the state with diagonal correlation tensor. It was recently shown that there are such unique local (stochastically reversible) filtering operations F_S, F_M applicable on a single copy of a qubit pair ($F_S^{\dagger}F_S \leq 1_S$ and $F_M^{\dagger}F_M \leq 1_M$) that transform (with a non-zero probability) any two-qubit mixed state into a state which is (i) diagonal in Bell basis and (ii) has the Bell factor $B'_{\max} \geq B_{\max}$ [10]. Since these Bell-diagonal states have the both local states maximally disordered the a-priori knowledges vanish. Thus – because the inequality (2) is satisfied also after the filtering – we can always saturate it with the upper bound given by $B'_{\max}$ just by an appropriate choice of the measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$ after the appropriate local filtering.

EXPERIMENT

We have verified inequality (2) experimentally for two Werner states of qubits, $p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\mathbb{1}$ (each qubit was represented by a polarization of a photon) [11]. The parameter of the first Werner state ($p_1 \approx 0.82$) has been chosen so that the state was entangled and violated Bell inequalities, the parameter of the second one ($p_2 \approx 0.45$) so that it was entangled but did not violate Bell inequalities. The scheme of our experimental setup is shown in Fig. 1. A krypton-ion cw laser (413.1 nm, 90 mW) is used to pump a 10-mm-long $LiIO_3$ nonlinear crystal cut for degenerate type-I parametric downconversion. We exploit the fact that the pairs of photons generated by spontaneous parametric downconversion (SPDC) manifest tight time correlations. In our setup the photons produced by SPDC have horizontal linear polarizations. Different linear-polarization states are prepared by means of half-wave plates ($\lambda/2$). The two photons impinge on two input ports of a beamsplitter (BS) forming a Hong-Ou-Mandel (HOM) interferometer [12]. A scanning mirror is used in one interferometer arm in order to balance the length of both arms, as indicated by an arrow in Fig. 1. A glass plate (GP), that introduces polarization dependent losses, serves to compensate a non-ideal splitting ratio of the beam-splitting cube (it is about 51:49 for vertical and 55:45 for horizontal polarization). The HOM interferometer enables us to prepare conditionally polarization singlet states (i.e., $|\Psi^-\rangle$ Bell states). The simplest theoretical model of the beamsplitter leads to the conclusion that if one fetches Bell states at the input the only one of them that results in a coincident detection at two different outputs of the beamsplitter is the singlet state $|\Psi^-\rangle$. However, in case of a "real" beam-splitting cube one must take into account that the two photons strike upon a beamsplitter in *opposite* directions. So, the mutual phase (at the interface plane) of the horizontal components of the electric-field vectors from the two opposite inputs is shifted by 180° just for *geometrical* reasons. Therefore it is the triplet state $|\Psi^+\rangle$ that leads to a coincident detection at different outputs. However, it is easy to change $|\Psi^+\rangle$ to $|\Psi^-\rangle$ by means of a half-wave plate placed in one output arm of the BS.

The mesurement block in each output arm consists of a half-wave plate and polarizing beamsplitter (PBS). It enables measurement in any linear-polarization basis. Behind the PBS the beams are filtered by cut-off filters and fed into

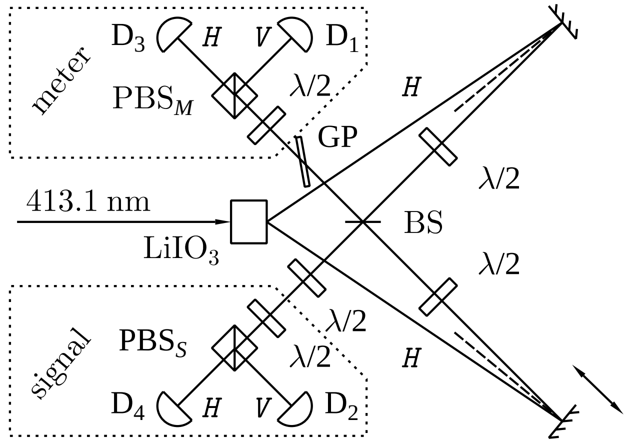

FIGURE 1. Experimental setup.

multi-mode optical fibers leading to detectors $D_1, \ldots, D_4$ (Perkin-Elmer single-photon counting modules; quantum efficiency $\eta \approx 50\%$, dark counts about $100\,\mathrm{s}^{-1}$).

The Werner states were prepared as a "mixture" of three kinds of inputs. First we measured coincidences with horizontal and vertical polarizations in the individual inputs of the HOM interferometer (measurement time for each point in the following graphs was 22 s), then we added the results of measurement with two horizontally polarized input photons (this measurement period took 10 s), and finally we measured with two vertically polarized input photons (13 s). The different times of measurement compensated the influence of a glass plate (GP) for the vertical-vertical and horizontal-horizontal input polarizations. The different values of parameter p were obtained changing the position of the scanning mirror. Namely, we have measured at $0\,\mu$m and $30\,\mu$m from the dip center.

The measurement Π_M on the "meter" qubit was represented by a measurement in different linear polarization bases parametrized by an angle ϑ: $\Pi_M = \equiv \{\Pi_M^+, \Pi_M^-\} = \{|\psi\rangle\langle\psi|, |\psi_\perp\rangle\langle\psi_\perp|\}$, where $|\psi\rangle = \cos\vartheta\,|H\rangle + \sin\vartheta\,|V\rangle$ and $|\psi_\perp\rangle = \sin\vartheta\,|H\rangle - \cos\vartheta\,|V\rangle$. The angle ϑ was set by a properly rotated half-wave plate. Similarly, two measurements on the "signal" qubit, Π_S and Π'_S, were represented by polarization measurements in two bases rotated by 45°: $\Pi_S = \{\Pi_S^+, \Pi_S^-\} \equiv \{|H\rangle\langle H|, |V\rangle\langle V|\}, \Pi'_S = \{\Pi'^+_S, \Pi'^-_S\} \equiv \{|X\rangle\langle X|, |Y\rangle\langle Y|\}$, where $|X\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and $|Y\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$. In practice we measured coincidence rates between outputs Π_M^+ and Π_S^+, between Π_M^+ and Π_S^-, etc. and the results were used for calculating values of the knowledge excesses $\Delta\mathbf{K}(\vartheta), \Delta\mathbf{K}'(\vartheta')$. The maximal violation of Bell inequalities, $B_{\max}$, was measured as in [4, 5] by estimating the correlation function from the measured data (coincidence rates).

The following graphs display our experimental results. Fig. 2 shows the sum $\Delta\mathbf{K}^2(\vartheta) + \Delta\mathbf{K}'^2(\vartheta')$ as a function of two angle variables for the Werner states with $p \approx 0.82$ (left graph) and $p \approx 0.45$ (right graph). These parameters were estimated from the best fit accordingly to the theoretical predictions for Werner states. The error bars show statistical errors. The accuracy of polarization-angle settings was about $\pm 1^\circ$. The maximal displayed value of the vertical axis determines the measured value of $(B_{\max}/2)^2$. In $p \approx 0.82$ (left graph) the maximal measured Bell factor is $B_{\max} = 2.36 \pm 0.02$ what is in a good agreement with the theoretical value for $p = 0.820 \pm 0.007$ that equals 2.319 ± 0.020. For Werner state with $p \approx 0.45$ (right graph) the corresponding measured maximal Bell factor is $B_{\max} = 1.32 \pm 0.02$ (theoretical value for $p = 0.450 \pm 0.008$ is 1.273 ± 0.023). As can be seen, for the both measured states the experiment has verified inequality (2).

CONCLUSION

The measurement on the one of two correlated particles give us a power of prediction of the measurement results on the other one. Of course, one can never predict exactly the results of two complementary measurements at once. However, knowing what kind of measurement we want to predict on "signal" particle, we can choose the optimal measurement on the "meter" particle. But there is still a fundamental limitation given by the sort and amount of correlations between the particles. Both these kinds of constraints are quantitatively expressed by our inequality.

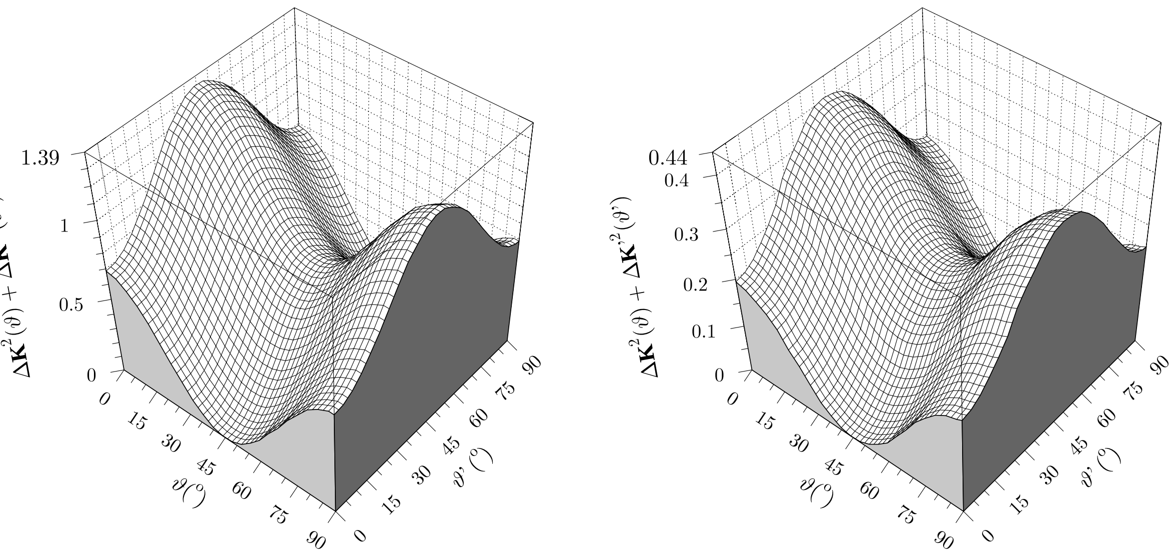

FIGURE 2. The measured values of the sum $\Delta\mathbf{K}^2(\vartheta) + \Delta\mathbf{K}'^2(\vartheta')$ as a function of two angle variables for the Werner states with $p \approx 0.82$ (left graph) and $p \approx 0.45$ (right graph). The maximal displayed values of the vertical axes show the measured values of $(B_{\max}/2)^2$.

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