How Quantum Correlations Enhance Prediction of Complementary Measurements

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If there are correlations between two qubits, then the results of the measurement on one of them can help to predict measurement results on the other one. It is an interesting question as to what can be predicted about the results of two complementary projective measurements on the first qubit. To quantify these predictions the complementary *knowledge excesses* are used. A nontrivial constraint restricting them is derived. For any mixed state and for arbitrary measurements the knowledge excesses are bounded by a factor depending only on the maximal violation of Bell's inequalities. This result is experimentally verified on two-photon Werner states prepared by means of spontaneous parametric down-conversion.

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Quantum correlations have attracted the attention of physicists since the early days of quantum mechanics. Einstein *et al.* used the features of quantum correlations in their argumentation against the completeness of quantum theory [1]. It was building on the fact that the result of any potential measurement on one subsystem of the properly chosen entangled pair can be predicted with certainty after the proper measurement on the other subsystem. Following this fact and a few "natural" assumptions they concluded that there must simultaneously exist "elements of reality" for two *complementary* observables. Later, it was shown that quantum mechanics predicts different values of certain correlations of measurement results than any local realistic theory. Inequalities, which have to be satisfied within the local realism, were derived by Bell [2]. The predictions of quantum mechanics have been convincingly experimentally confirmed using, e.g., pairs of photons entangled in polarizations [3]. In this Letter, we analyze quantitatively how the correlations between the qubits prepared in a general mixed state enhance our ability to predict the results of complementary projective measurements on one qubit when we know the measurement results on the other one. This enhancement can be described by the quantity that we will call complementary knowledge excess. We have derived a nontrivial bound on the knowledge excesses which is determined only by the maximal violation of Bell inequalities [4]. An experimental test of this restriction was performed using a two-photon Werner state prepared by means of spontaneous parametric down-conversion.

We assume a general mixed state ρ_{SM} of a "signal" qubit S and a "meter" qubit M. Performing two projective measurements Π_M, Π'_M on qubit M, the prediction of the results of mutually complementary measurements Π_S, Π'_S on qubit S can be improved. Complementarity of measurements on a qubit means that $\mathrm{Tr}\Pi_{Si}\Pi'_{Sj} = 1/2$ for any $i, j = 0, 1$ (Π_{Si}, Π'_{Si} are corresponding projectors). Assuming $\Pi_{S0} = |\Psi\rangle_S\langle\Psi|$, $\Pi_{S1} = |\Psi^\perp\rangle_S\langle\Psi^\perp|$, we can expand the state ρ_{SM} in the form $\rho_{SM} = w|\Psi\rangle_S\langle\Psi| \otimes \rho_M + w^\perp|\Psi^\perp\rangle_S\langle\Psi^\perp| \otimes \rho_M^\perp + \sqrt{ww^\perp}(|\Psi\rangle_S\langle\Psi^\perp| \otimes \chi_M + \mathrm{h.c.})$, where $0 \le w, w^\perp \le 1$, $w + w^\perp = 1$ and the meter operators $\rho_M, \rho_M^\perp, \chi_M$ depend on the choice of the measurement Π_S. In order to predict the result of the measurement Π_S one needs to discriminate between the mixed states ρ_M and $\rho_M^\perp$ by a projective two-component measurement $\Pi_M = \{\Pi_{M0}, \Pi_{M1}\}$ on the qubit M. Using maximum likelihood estimation strategy, we can guess for each detection event the most likely result of the measurement Π_S. Our knowledge can be quantified as the fractional excess of the right guesses over wrong guesses in many such experiments repeated under identical conditions [5]. Using our expansion of ρ_{SM}, the total knowledge is $\mathbf{K}(\Pi_M \to \Pi_S) = \sum_i|\mathrm{Tr}_M\Pi_{Mi}(w\rho_M - w^\perp\rho_M^\perp)|$, whereas without the measurement Π_M, the knowledge is $\mathbf{P}(\Pi_S) = |w - w^\perp|$. The largest value of knowledge over all Π_M was introduced as distinguishability $\mathbf{D}(\Pi_S) = \mathrm{Tr}_M|(w\rho_M - w^\perp\rho_M^\perp)|$. Further, we define a knowledge excess $\Delta\mathbf{K}(\Pi_M \to \Pi_S) = \mathbf{K}(\Pi_M \to \Pi_S) - \mathbf{P}(\Pi_S)$, where $0 \le \Delta\mathbf{K}(\Pi_M \to \Pi_S) \le 1$. It quantifies only that amount of the knowledge which exceeds the *a priori* knowledge $\mathbf{P}(\Pi_S)$. The largest $\Delta\mathbf{K}(\Pi_M \to \Pi_S)$ over all Π_M can be considered as a distinguishability excess $\Delta\mathbf{D}(\Pi_S)$. Thus $0 \le \Delta\mathbf{K}(\Pi_M \to \Pi_S) \le \Delta\mathbf{D}(\Pi_S)$. Analogical quantities $\Delta\mathbf{K}(\Pi'_M \to \Pi'_S)$ and $\Delta\mathbf{D}(\Pi'_S)$ can be defined for the complementary measurement Π'_S.

Intuitively, for a given mixed state ρ_{SM} of a two-qubit system the knowledge excesses are somehow restricted by the properties of the state. To derive a quantitative constraint, we will use the following expansion of the state: $\rho_{SM} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \sum_{l=1}^3 m_l\sigma_l + \sum_{l=1}^3 n_l\sigma_l \otimes \mathbb{1} + \sum_{k,l=1}^3 t_{kl}\sigma_k \otimes \sigma_l)$, where $\mathbb{1}$ stands for the identity operator; m_l, n_l are vectors in R^3; σ_l, $l = 1, 2, 3$ are the standard Pauli operators. The coefficients t_{kl} form a real correlation matrix T and the vectors m_l and n_l determine the local states $\rho_S = \frac{1}{2}(\mathbb{1} + \sum_l n_l\sigma_l)$, $\rho_M = \frac{1}{2}(\mathbb{1} + \sum_l m_l\sigma_l)$. There

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is a subset of the states $\bar{\rho}_{SM}$ having a diagonal correlation tensor $\bar{T} = \mathrm{diag}(\bar{t}_{33}, \bar{t}_{11}, \bar{t}_{22})$, with the following property $\bar{t}_{33}^2 \geq \bar{t}_{11}^2, \bar{t}_{22}^2$. The local states are determined by the corresponding vectors $\bar{m}_l$ and $\bar{n}_l$. Any mixed state ρ_{SM} can be uniquely converted to a state $\bar{\rho}_{SM}$ using appropriate local unitary operations [6]. Thus, just two orderings of the diagonal elements, $\bar{t}_{11}^2 \geq \bar{t}_{22}^2$ or $\bar{t}_{11}^2 \leq \bar{t}_{22}^2$, remain to be discussed.

Let us suppose the measurements $\bar{\Pi}_S$ and $\bar{\Pi}_M$ are constructed from projectors to the vectors of the local bases in which $|\bar{t}_{33}|$ is maximal. Then $\Delta\mathbf{D}(\bar{\Pi}_S) = \Delta\mathbf{K}(\bar{\Pi}_M \to \bar{\Pi}_S) = \max(0, |\bar{t}_{33}| - |\bar{n}_3|)$; for this choice of measurements $\Delta\mathbf{K}$ gets its maximal value over all Π_M. Let us suppose the other pair of measurements. If $\bar{t}_{11}^2 \geq \bar{t}_{22}^2$, let $\bar{\Pi}'_S$ and $\bar{\Pi}'_M$ be related to the bases in which $|\bar{t}_{11}|$ is maximal, then $\Delta\mathbf{D}(\bar{\Pi}'_S) = \Delta\mathbf{K}(\bar{\Pi}'_M \to \bar{\Pi}'_S) = \max(0, |\bar{t}_{11}| - |\bar{n}_1|)$. Simultaneously, we express the violation of any Bell's inequalities employing the criterion from Ref. [4]: A state $\bar{\rho}_{SM}$ violates Bell's inequalities if its maximal Bell factor $B_{\max} = 2\sqrt{\bar{t}_{11}^2 + \bar{t}_{33}^2}$ lies in the interval $(2, 2\sqrt{2}]$. Analogical results can be derived if $\bar{t}_{11}^2 \leq \bar{t}_{22}^2$: $\Delta\mathbf{D}(\bar{\Pi}'_S) = \max(0, |\bar{t}_{22}| - |\bar{n}_2|)$. The maximal Bell factor is then $B_{\max} = 2\sqrt{\bar{t}_{22}^2 + \bar{t}_{33}^2}$. Finally we obtain an inequality $\Delta\mathbf{D}^2(\bar{\Pi}_S) + \Delta\mathbf{D}^2(\bar{\Pi}'_S) \leq (B_{\max}/2)^2$ valid for an arbitrary state $\bar{\rho}_{SM}$. The equality occurs for states with zero *a priori* knowledge. For such states a nonzero knowledge can be obtained only though the measurement on M.

Now we generalize these results to any state ρ_{SM} as well as for arbitrary measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$, where Π_S, Π'_S are complementary measurements. As pointed out, any mixed two-qubit state can be uniquely prepared from some state $\bar{\rho}_{SM}$ (of a special form discussed above) by appropriate local unitary transformations U_S, U_M acting on qubits S and M, respectively. Further, the transformation of the above chosen measurements $\bar{\Pi}_S$ and $\bar{\Pi}'_S$ to arbitrary (but still complementary) measurements Π_S and Π'_S corresponds effectively to the extra local unitary transformation U_Π acting on the qubit S. Since distinguishabilities $\Delta\mathbf{D}(\Pi_S)$ and $\Delta\mathbf{D}(\Pi'_S)$ are invariant under any local unitary transformation on the qubit M, it is sufficient to take into account only a joint unitary transformation $\tilde{U}_S = U_\Pi U_S$ acting on qubit S. For any unitary transformations U there is a unique rotation O such that $U(\vec{n}\cdot\vec{\sigma})U^\dagger = (O\vec{n})\cdot\vec{\sigma}$. If a state $\bar{\rho}_{SM}$ with diagonal $\bar{T}$ is subjected to the $U_S \otimes U_M$ transformation its correlation matrix transforms as follows $T = O_S\bar{T}O_M^\dagger$ [6]. Thus a joint unitary transformation $\tilde{U}_S$ can be represented as a transformation of the correlation tensor $T = O_S\bar{T}$, where O_S is a matrix of rotation in R^3 space.

First, we will explicitly calculate $\Delta\mathbf{D}(\Pi_S)$ and $\Delta\mathbf{D}(\Pi'_S)$ for any mixed state using the transformation $T = O_S\bar{T}$. Assuming $\bar{t}_{11}^2 \geq \bar{t}_{22}^2$ we obtain $\Delta\mathbf{D}(\Pi_S) = \max(0, \sqrt{t_{33}^2 + t_{32}^2 + t_{31}^2} - |n_3|)$ and $\Delta\mathbf{D}(\Pi'_S) = \max(0, \sqrt{t_{11}^2 + t_{12}^2 + t_{13}^2} - |n_1|)$. Then we straightforwardly get $\Delta\mathbf{D}^2(\Pi_S) + \Delta\mathbf{D}^2(\Pi'_S) \leq (B_{\max}/2)^2$. By analogous calculations we obtain the same result for $\bar{t}_{11}^2 \leq \bar{t}_{22}^2$. Finally, since $\Delta\mathbf{K}(\Pi_M \to \Pi_S) \leq \Delta\mathbf{D}(\Pi_S)$ and $\Delta\mathbf{K}(\Pi'_M \to \Pi'_S) \leq \Delta\mathbf{D}(\Pi'_S)$ we can conclude that

$$\Delta\mathbf{K}^2(\Pi_M \to \Pi_S) + \Delta\mathbf{K}^2(\Pi'_M \to \Pi'_S) \leq \left(\frac{B_{\max}}{2}\right)^2. \quad (1)$$

Thus the maximal Bell factor represents a nontrivial bound on the sum of the squares of knowledge excesses which can be extracted from a pair of measurements on the meter qubit. Assuming $\Pi_M = \Pi'_M$ we can also derive an inequality analogous to that given in Ref. [5]: $\Delta\mathbf{K}^2(\Pi_M \to \Pi_S) + \Delta\mathbf{K}^2(\Pi_M \to \Pi'_S) \leq 1$. Our analysis shows that for $\Pi_M \neq \Pi'_M$ the unit value on the right-hand side may be overstepped. Note also that $(B_{\max}/2)^2 > 1$ only if the state violates Bell inequalities. For details of the proofs see Ref. [7].

A natural question is how inequality (1) can be saturated. For the class of states with vanishing *a priori* knowledge for any measurements Π_S, Π'_S it can be saturated just by the appropriate choice of measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$. In fact, it corresponds to the transformation of the given state to the state with diagonal correlation tensor. It was recently shown that there are such unique local filtering operations applicable on a single copy of a qubit pair that transform any two-qubit mixed state into a state which is (i) diagonal in Bell basis and (ii) has the Bell factor $B'_{\max} \geq B_{\max}$ [8]. Since these Bell-diagonal states have the both local states maximally disordered the *a priori* knowledge vanishes. Thus—because the inequality (1) is satisfied also after the filtering—we can always saturate it with the upper bound given by $B'_{\max}$ just by an appropriate choice of the measurements $\Pi_S, \Pi'_S, \Pi_M, \Pi'_M$ after the appropriate local filtering.

We have verified inequality (1) experimentally for two Werner states of qubits, $p|\Psi^-\rangle\langle\Psi^-| + [(1-p)/4]\mathbb{1}$ (each qubit was represented by a polarization of a photon) [9]. The parameter of the first Werner state ($p_1 \approx 0.82$) has been chosen so that the state was entangled and violated Bell inequalities, the parameter of the second one ($p_2 \approx 0.45$) so that it was entangled but did not violate Bell inequalities. The scheme of our experimental setup is shown in Fig. 1. A krypton-ion cw laser (413.1 nm, 90 mW) is used to pump a 10-mm-long $LiIO_3$ nonlinear crystal cut for degenerate type-I spontaneous parametric down-conversion. The generated photons have horizontal linear polarizations. Different linear-polarization states are prepared by means of half-wave plates ($\lambda/2$). The two photons impinge on two input ports of a beam splitter (BS) forming a Hong-Ou-Mandel (HOM) interferometer

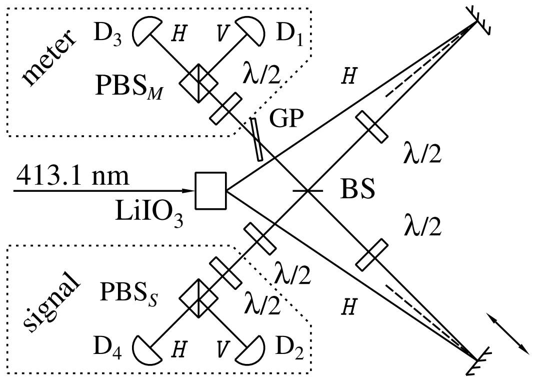

FIG. 1. Experimental setup.

[10]. A scanning mirror is used in one interferometer arm in order to balance the length of both arms. A glass plate (GP), that introduces polarization dependent losses, serves to compensate a nonideal splitting ratio of the beam-splitting cube. HOM interferometer enables us to prepare conditionally polarization singlet states (i.e., $|\Psi^-\rangle$ Bell states). The simplest theoretical model of the beam splitter leads to the conclusion that if one fetches Bell states at the input the only one of them that results in a coincident detection at two different outputs of the beam splitter is the singlet state $|\Psi^-\rangle$. However, in the case of a "real" beam-splitting cube one must take into account that the two photons strike upon a beam splitter in *opposite* directions. Therefore, it is the triplet state $|\Psi^+\rangle$ that leads to a coincident detection at different outputs. However, it is easy to change $|\Psi^+\rangle$ to $|\Psi^-\rangle$ by means of a half-wave plate placed in one output arm of the BS. The measurement block in each output arm consists of a half-wave plate and polarizing beam splitter (PBS). It enables measurement in any linear-polarization basis. Behind the PBS the beams are filtered by cutoff filters and fed into multimode optical fibers leading to detectors $D_1, \ldots, D_4$ (Perkin-Elmer single-photon counting modules).

The Werner states were prepared as a mixture of three kinds of states. First we measured coincidences with horizontal and vertical polarizations in the individual inputs of HOM interferometer (measurement time for each point in the following graphs was 22 s), then we added the results of same measurement but with two horizontally polarized input photons (this measurement period took 10 s), and finally we measured with two vertically polarized input photons (13 s). The different times of measurement compensated the influence of a GP for the vertical-vertical and horizontal-horizontal input polarizations. The different values of parameter p were obtained changing the position of the scanning mirror. Namely, we have measured at 0 and 30 μm from the dip center.

The measurement Π_M on the meter qubit was represented by a measurement in different linear-polarization bases parametrized by an angle ϑ: $\Pi_M = \{\Pi_M^+, \Pi_M^-\} \equiv \{|\psi\rangle\langle\psi|, |\psi_\perp\rangle\langle\psi_\perp|\}$, where $|\psi\rangle = \cos\vartheta|H\rangle + \sin\vartheta|V\rangle$ and $|\psi_\perp\rangle = \sin\vartheta|H\rangle - \cos\vartheta|V\rangle$. The angle ϑ was set by a properly rotated half-wave plate. Similarly, two measurements on the signal qubit, Π_S and Π'_S, were represented by polarization measurements in two bases rotated by 45°: $\Pi_S = \{\Pi_S^+, \Pi_S^-\} \equiv \{|H\rangle\langle H|, |V\rangle \times \langle V|\}$, $\Pi'_S = \{\Pi_S'^+, \Pi_S'^-\} \equiv \{|X\rangle\langle X|, |Y\rangle\langle Y|\}$, where $|X\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and $|Y\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$. In practice we measured coincidence rates between outputs Π_M^+ and Π_S^+ (it is denoted C^{++}), between Π_M^+ and Π_S^- (it is denoted C^{+-}), etc. (the first sign concerns the M qubit, the second one the S qubit). Then the knowledge $\mathbf{K}(\vartheta) = |\mathrm{Tr}_{MS}\Pi_M^+(\Pi_S^+ - \Pi_S^-)\rho| + |\mathrm{Tr}_{MS}\Pi_M^-(\Pi_S^+ - \Pi_S^-)\rho|$ can be calculated from measured rates as follows: $(|C^{++} - C^{+-}| + |C^{-+} - C^{--}|)/N$, where $N = C^{++} + C^{+-} + C^{-+} + C^{--}$. Analogously the *a priori* knowledge $\mathbf{P} = |\mathrm{Tr}_{MS}(\Pi_S^+ - \Pi_S^-)\rho|$ can be obtained as $|(C^{++} + C^{-+}) - (C^{+-} + C^{--})|/N$. The knowledge excess is given as $\Delta\mathbf{K}(\vartheta) = \mathbf{K}(\vartheta) - \mathbf{P}$. The quantities $\mathbf{K}'(\vartheta)$, $\mathbf{P}'$, and $\Delta\mathbf{K}'(\vartheta)$ are obtained in the same way just with Π'_S instead of Π_S. The maximal violation of Bell inequalities, $B_{\max}$, can be obtained by measuring correlation functions for two different polarization bases on each side. Namely, for the Werner states one can choose these bases rotated by 22.5° and 67.5° (with respect to the vertical axis) on the one side and 45° and 0° on the other side: $B_{\max} = |C(22.5°, 45°) + C(67.5°, 45°) + C(22.5°, 0°) - C(67.5°, 0°)|$, where the correlation function $C(\vartheta_1, \vartheta_2)$ is estimated from the measured data as $(C^{++} + C^{--} - C^{+-} - C^{-+})/N$. Let us note that for Werner states the theoretical predictions of regarded quantities read $\mathbf{K} = p|\cos(2\vartheta)|$, $\mathbf{K}' = p|\sin(2\vartheta)|$, $\mathbf{P} = \mathbf{P}' = 0$, $B_{\max} = p2\sqrt{2}$. Clearly, maximal value of $\Delta\mathbf{K}^2(\vartheta) + \Delta\mathbf{K}'^2(\vartheta')$ should appear for $\vartheta = 0°$ (and 90°), $\vartheta' = 45°$.

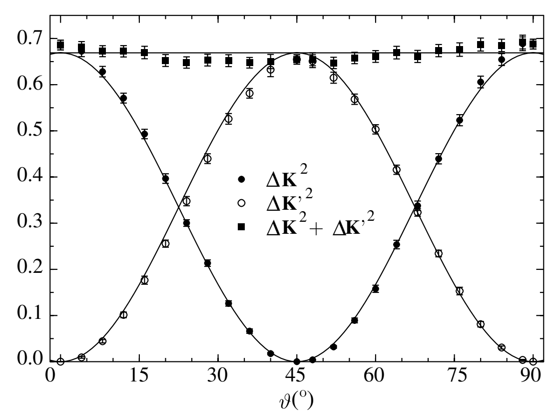

FIG. 2. The squares of the knowledge excesses and their sum measured for the Werner state with $p \approx 0.82$. Symbols show experimental values, full lines theoretical predictions (for $p = 0.82$).

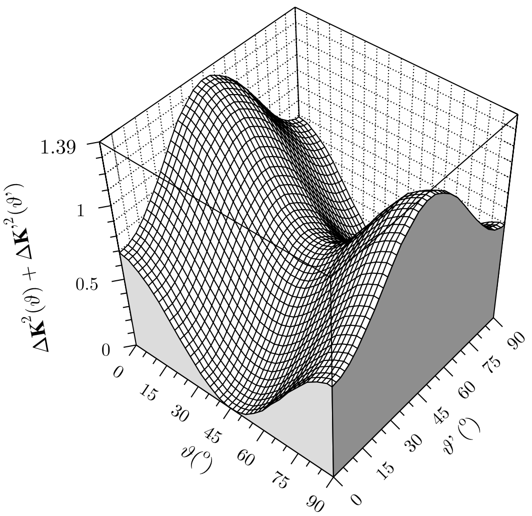

FIG. 3. The measured values of the sum $\Delta\mathbf{K}^2(\vartheta) + \Delta\mathbf{K}'^2(\vartheta')$ as a function of two angle variables for the Werner state with $p \approx 0.82$. The maximal displayed value of the vertical axis shows the measured value of $(B_{\max}/2)^2$.

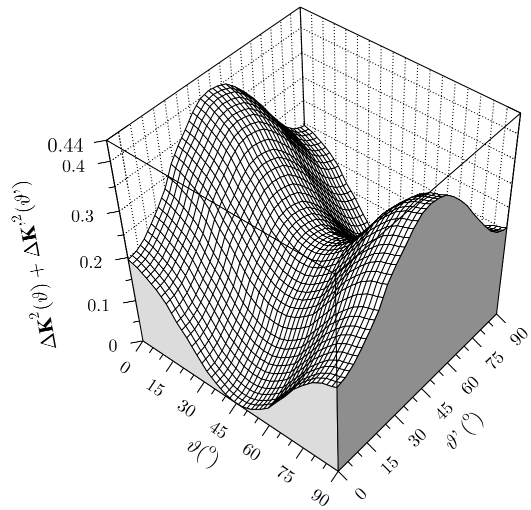

FIG. 4. The same kind of results as in Fig. 3 but for $p \approx 0.45$.

The following graphs display our experimental results. In Fig. 2 there are plotted the squares of the knowledge excesses $\Delta\mathbf{K}^2(\vartheta)$, $\Delta\mathbf{K}'^2(\vartheta)$ and their sum measured for the Werner state with parameter $p \approx 0.82$ (this parameter was estimated from the best fit accordingly to the theoretical predictions for Werner states). The error bars show statistical errors. The accuracy of polarization-angle settings was about $\pm 1°$. Figure 3 shows the sum $\Delta\mathbf{K}^2(\vartheta) + \Delta\mathbf{K}'^2(\vartheta')$ as a function of two angle variables for the same Werner state. The maximal displayed value of the vertical axis determines the measured value of $(B_{\max}/2)^2$. The maximal measured Bell factor is $B_{\max} = 2.36 \pm 0.02$ what is in a good agreement with the theoretical value for $p = 0.818 \pm 0.007$ that equals 2.314 ± 0.020. The same kind of measurement is presented in Fig. 4 but now for the Werner state with $p \approx 0.45$. The corresponding measured maximal Bell factor is $B_{\max} = 1.32 \pm 0.02$ (theoretical value for $p = 0.453 \pm 0.008$ is 1.281 ± 0.023). As can be seen, for the both measured states the experiment has verified inequality (1).

The measurement on the one of two correlated particles give us the power of prediction of the measurement results on the other one. Of course, one can never predict exactly the results of two complementary measurements at once. However, knowing what kind of measurement we want to predict on signal particle, we can choose the optimal measurement on the meter particle. But there is still a fundamental limitation given by the sort and amount of correlations between the particles. Both of these kinds of constraints are quantitatively expressed by our inequality (1). The limitation stemming from mutual correlation of particles manifests itself by the maximal Bell factor appearing in the inequality. We have proved this inequality theoretically as well as tested it experimentally.

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