Probabilistic quantum multimeters

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We propose quantum devices that can realize probabilistically different projective measurements on a qubit. The desired measurement basis is selected by the quantum state of a program register. First we analyze the phase-covariant multimeters for a large class of program states and then the universal multimeters for a special choice of program. In both cases we start with deterministic but error-prone devices and then proceed to devices that never make a mistake but from time to time give an inconclusive result. These multimeters are optimized (for a given type of program) with respect to the minimum probability of an inconclusive result. This concept is further generalized to multimeters that minimize the error rate for a given probability of an inconclusive result (or vice versa). Finally, we propose a generalization for qudits.

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I. INTRODUCTION

Programmable quantum multimeters are devices that can realize any desired generalized quantum measurement from a chosen set (either exactly or approximately) [1,2]. Their main feature is that the particular positive operator valued measure (POVM) is selected by the quantum state of a "program register" (quantum software). In this sense they are analogous to universal quantum processors [3–6]. The multimeter itself is represented by a *fixed* joint POVM on the data and program systems together (see Fig. 1). Each outcome of this POVM is associated with one outcome of the "programmed" POVM on the data alone. From the mathematical point of view the realization of a particular quantum multimeter is equivalent to the optimal discrimination of certain mixed states. A different kind of quantum multimeter that can be programmed to evaluate the expectation value of any operator has been introduced in Ref. [7]. In addition to quantum multimeters, other devices whose operation is based on the joint measurement on two different registers have been proposed recently. A universal quantum matching machine that allows a decision as to which template state is closest to the input feature state was analyzed in [8]. The problem of comparison of quantum states was studied in [9]. So-called universal quantum detectors were considered in [10].

Programmable quantum multimeters [1,2] and gate arrays [3–6] are in some sense similar to classical computers. In both cases a single fixed machine (hardware) can be used to perform many different operations on the state of the data register, and the operation is controlled by the state of the program register. In particular, programmable quantum multimeters can be considered as a model of measurement-based quantum computation. Possible cryptographic applications of programmable quantum machines have also been considered in the literature [4]. For instance, one can imagine a situation where one party (Alice) provides a program while another party (Bob) actually uses the program to perform a measurement with the programmable multimeter. As we shall see below, the programs of the multimeters are nonorthogonal, yet they allow for a perfect (albeit probabilistic) operation of these devices. Since nonorthogonal program states are used, Bob cannot learn with certainty what the measurement was. All these considerations strongly suggest that the programmable quantum devices could play an important role in quantum information processing.

In this paper, we will describe programmable quantum devices that can accomplish von Neumann measurements on a single qubit. However, it is impossible to perfectly encode arbitrary projective measurement on a qubit into a state in finite-dimensional Hilbert space [1]. The proof of this theorem is similar to the proof that it is impossible to encode an arbitrary unitary operation (acting on a finite-dimensional Hilbert space) into a state of a finite-dimensional quantum system [3]. Briefly, one can show that any two program states that perfectly encode two different measurement bases must be mutually orthogonal. Nevertheless, it is still possible to encode POVMs that represent, in a certain sense, the best approximation of the required projective measurements.

A specific way of approximation of projective measurements is a "probabilistic" measurement that allows for some inconclusive results. In this case, instead of a two-component projective measurement, one has a three-component POVM and the third outcome corresponds to the inconclusive result. The natural request is to minimize the error rate at the first two outcomes. As a limit case it is possible to get an error-free operation (however, with a nonzero probability of an inconclusive result); such a multimeter performs exact projective measurements but with a probability of success lower than 1. Such a device is conceptually analogous to probabilistic programmable quantum gates [3–5]. The other boundary case is an ambiguous multimeter without inconclusive results [2].

One possible way to implement quantum multimeters is to exploit programmable gate arrays. A projective measurement in any basis $\{|\psi_j\rangle\}_{j=1}^{d}$ can be performed as a sequence of a (programmed) unitary operation that maps the measurement basis onto the fixed computational basis $\{|j\rangle\}_{j=1}^{d}$, followed by a measurement in the computational basis. However, the approach based on programmable gate arrays need not be optimal. Also, most programmable gate arrays considered in the literature are probabilistic and they involve complicated entangled multiqubit programs. In contrast, we consider both deterministic and probabilistic multimeters with simple programs in product states. Interestingly, Vidal, Masanes, and Cirac (VMC) proposed a probabilistic gate array that can perform any rotation of a qubit about the z axis of the Poincaré sphere and employs a product-state program [4]. We will discuss the links of the VMC gate with phase-covariant multimeters below.

Our present article is organized as follows. In Sec. II we start with the analysis of phase-covariant multimeters that can perform von Neumann measurement on a single qubit in any basis located on the equator of the Bloch sphere. First we discuss deterministic devices (no inconclusive results but errors may appear), then error-free probabilistic devices (no errors but inconclusive results may appear), and finally general multimeters with a given fraction of inconclusive results optimized with respect to a minimal error rate. Moreover, we give a brief discussion of a possible optical implementation of the simplest phase-covariant multimeter with a single-qubit program. At the end of the section we construct the optimal phase-covariant multimeter for the VMC program and we prove that the VMC gate followed by measurement in the fixed basis $(|0\rangle \pm |1\rangle)/\sqrt{2}$ in fact represents the optimal multimeter for the VMC program. In Sec. II we also introduce and explain in detail all necessary mathematical tools.

Further, in Sec. III we study universal multimeters that can accomplish *any* von Neumann measurement on a single qubit. We confine our investigation to a program consisting of the two basis states. Again, we start with deterministic devices, continue with error-free multimeters, and finally proceed to apparatuses with a given fraction of inconclusive results. Section IV is devoted to probabilistic error-free universal multimeters that can accomplish *any* projective measurement on a *qudit*. Section V concludes the paper with a short summary.

II. PHASE-COVARIANT MULTIMETERS

In this section we will consider multimeters that should perform von Neumann measurement on a single qubit in any basis $\{|\psi_+\rangle, |\psi_-\rangle\}$ located on the equator of the Bloch sphere,

$$|\psi_\pm(\phi)\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\phi}|1\rangle), \tag{1}$$

where $\phi \in [0, 2\pi]$ is arbitrary. The multimeter should be phase covariant, i.e., it should operate equally well for all phases ϕ. To simplify notation, we shall not usually display the dependence of the basis states on ϕ explicitly in what follows. Generally, the design of the optimal multimeter should involve the optimization of both the dependence of the program on the measurement basis and the fixed joint measurement on the program and data registers. However, this is a very hard problem that we will not attempt to solve in its generality. Instead, we will design an optimal multimeter for a particular simple and natural choice of program; namely, as in [2], we assume that the program of the multimeter $|\Psi\rangle_p$ which determines the measurement basis consists

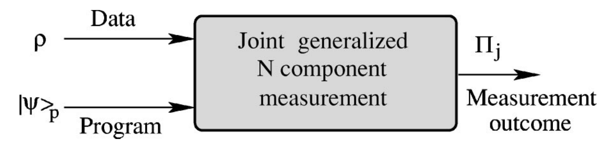

FIG. 1. Schematic drawing of a quantum multimeter. The effective measurement performed on the data state ρ is selected by the quantum state of the program register $|\Psi\rangle_p$. The multimeter itself carries out a *fixed* joint generalized measurement on data and program states which is described by a POVM $\{\Pi_j\}$.

of N copies of the basis state $|\psi_+\rangle$, $|\Psi\rangle_p = |\psi_+\rangle^{\otimes N}$. Since we have restricted ourselves to the bases (1), the state $|\psi_-\rangle$ can be obtained from $|\psi_+\rangle$ via unitary transformation,

$$|\psi_-\rangle = \sigma_z |\psi_+\rangle, \tag{2}$$

where σ_z denotes the Pauli matrix. This implies that all programs of the form $|\psi_+\rangle^{\otimes j}|\psi_-\rangle^{\otimes N-j}$ are equivalent to the program $|\psi_+\rangle^{\otimes N}$ because these programs are related via a *fixed* unitary $U = \mathbb{1}^{\otimes j} \otimes \sigma_z^{\otimes N-j}$. First, we shall derive the optimal deterministic multimeter, which always yields an outcome, but errors may occur. Then we shall consider a probabilistic multimeter that conditionally realizes exactly the von Neumann measurement in the basis (1), but at the expense of some fraction of inconclusive results. The deterministic and unambiguous multimeters are two extremal cases from a whole class of optimal multimeters that are designed such that the probability of a correct outcome for measurements on basis states is maximized for a fixed fraction of inconclusive results. Such generalized multimeters will also be studied in this section.

A. Deterministic multimeter

The multimeter is a device that performs a joint generalized measurement described by the POVM $\{\Pi_j\}$ on the data and program registers (see Fig. 1). This *fixed* joint measurement on the data and program can also be interpreted as an effective measurement on the data register, which is described by the POVM π_j and depends on the program via

$$\pi_j = \mathrm{Tr}_p[(\mathbb{1}_d \otimes |\Psi\rangle_p\langle\Psi|)\Pi_j], \tag{3}$$

where the subscripts "d" and "p" denote the data and program states, respectively, and $\mathbb{1}$ is the identity operator. The deterministic single-qubit multimeter is fully characterized by a two-component POVM $\{\Pi_+, \Pi_-\}$. Upon obtaining the outcome $\Pi_+(\Pi_-)$ one guesses the state of the data register to be $|\psi_+\rangle(|\psi_-\rangle)$. Ideally,

$$\pi_\pm = |\psi_\pm\rangle\langle\psi_\pm| \tag{4}$$

should hold, but this cannot be achieved for all ϕ with a finite-dimensional program.

The performance of the multimeter is quantified by the probability P_S that the measurement yields the correct outcome when the data register is prepared in the basis state $|\psi_+\rangle$ or $|\psi_-\rangle$ with probability $1/2$ each. For each particular phase ϕ we thus have

$$P_S(\phi) = \frac{1}{2}\mathrm{Tr}[\Pi_+ \psi_+(\phi) \otimes \psi_+^{\otimes N}(\phi)] + \frac{1}{2}\mathrm{Tr}[\Pi_- \psi_-(\phi) \otimes \psi_+^{\otimes N}(\phi)], \tag{5}$$

where $\psi_\pm = |\psi_\pm\rangle\langle\psi_\pm|$. Assuming a homogeneous *a priori* distribution of the angle ϕ, we define the average success rate as

$$P_S = \int_0^{2\pi} P_S(\phi)\frac{d\phi}{2\pi}. \tag{6}$$

We define the optimal deterministic multimeter for the program $|\psi_+\rangle^{\otimes N}$ as the multimeter that maximizes P_S. The choice of P_S as the figure of merit is strongly supported by the observation that P_S can be interpreted as the average *fidelity* of the multimeter. Consider the effective POVM on the data qubit $\{\pi_+(\phi), \pi_-(\phi)\}$ for some particular phase ϕ. It is natural to define the fidelity of this POVM with respect to the projective measurement in the basis $|\psi_\pm(\phi)\rangle$ as follows:

$$F(\phi) = \frac{1}{2}\langle\psi_+(\phi)|\pi_+(\phi)|\psi_+(\phi)\rangle + \frac{1}{2}\langle\psi_-(\phi)|\pi_-(\phi)|\psi_-(\phi)\rangle.$$

It is easy to see that the average fidelity $F = (1/2\pi)\int_0^{2\pi} F(\phi)d\phi$ coincides with the average success rate (6). Clearly, $F \leqslant 1$ and $F=1$ if and only if Eq. (4) holds for all ϕ (maybe except for a set of measure zero).

To simplify the notation we introduce the symbol $C_{N,k}$ for the binomial coefficient,

$$C_{N,k} = \binom{N}{k}. \tag{7}$$

On inserting the formula for $P_S(\phi)$ into Eq. (6) and carrying out the integration over ϕ we find that

$$P_S = \frac{1}{2}(\mathrm{Tr}[\Pi_+ R_+] + \mathrm{Tr}[\Pi_- R_-]), \tag{8}$$

where the two positive semidefinite operators $R_\pm$ read

$$R_+ = \frac{1}{2^{N+1}}\sum_{k=1}^{N} C_{N+1,k}|\varphi_{N,k}^+\rangle\langle\varphi_{N,k}^+| + \frac{1}{2^{N+1}}X,$$

$$R_- = \frac{1}{2^{N+1}}\sum_{k=1}^{N} C_{N+1,k}|\varphi_{N,k}^-\rangle\langle\varphi_{N,k}^-| + \frac{1}{2^{N+1}}X.$$

Here,

$$|\varphi_{N,k}^\pm\rangle = \sqrt{1-B_{N,k}}|0\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}} \pm \sqrt{B_{N,k}}|1\rangle_{\mathrm{d}}|N,k-1\rangle_{\mathrm{p}}, \tag{9}$$

with $B_{N,k}=k/(N+1)$. The operator X that is common to R_+ and R_- is given by

$$X = |0\rangle_{\mathrm{d}}\langle 0| \otimes |N,0\rangle_{\mathrm{p}}\langle N,0| + |1\rangle_{\mathrm{d}}\langle 1| \otimes |N,N\rangle_{\mathrm{p}}\langle N,N|, \tag{10}$$

and $|N,k\rangle$ denotes a normalized totally symmetric state of N qubits with k qubits in state $|1\rangle$ and $N-k$ qubits in state $|0\rangle$.

It follows from Eq. (8) that the optimal deterministic multimeter is the one that optimally discriminates between two mixed states R_+ and R_-. This problem has been analyzed by Helstrom [11], who showed that the maximal achievable success rate is

$$P_{S,\max} = \frac{1}{2} + \frac{1}{4}\mathrm{Tr}|R_+ - R_-|, \tag{11}$$

and the optimal POVM is given by projectors onto the subspaces spanned by the eigenstates of $\Delta R = R_+ - R_-$ with positive and negative eigenvalues, respectively. If some of the eigenvalues of ΔR are zero, then the projectors can be freely added to either Π_+ or Π_-.

In the basis $|0\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}}$, $|1\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}}$, the matrix ΔR is block diagonal and its eigenvalues and eigenstates can easily be determined. Since $\mathrm{Tr}|\Delta R|$ is equal to the sum of absolute values of the eigenvalues of ΔR, we find after simple algebra that

$$P_{S,\max} = \frac{1}{2} + \frac{1}{2^{N+1}}\sum_{k=1}^{N}\sqrt{\binom{N}{k}\binom{N}{k-1}}. \tag{12}$$

Interestingly enough, $P_{S,\max}$ is equal to the optimal fidelity of estimation of $|\psi_+(\phi)\rangle$ from N copies of $|\psi_+(\phi)\rangle$ [12]. So one possible implementation of the optimal deterministic phase-covariant multimeter with program $|\psi_+(\phi)\rangle^{\otimes N}$ would be to first carry out the optimal estimation of $|\psi_+(\phi)\rangle$ and then measure the data qubit in the basis spanned by the estimated state and its orthogonal counterpart. Instead, one could also perform a joint generalized measurement on data and program registers. The two elements of the optimal POVM that maximizes P_S are given by

$$\Pi_\pm = \sum_{k=1}^{N}|\Pi_{N,k}^\pm\rangle\langle\Pi_{N,k}^\pm| + \frac{1}{2}X, \tag{13}$$

where

$$|\Pi_{N,k}^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}} \pm |1\rangle_{\mathrm{d}}|N,k-1\rangle_{\mathrm{p}}). \tag{14}$$

The effective POVM on the data register (3) can be expressed as

$$\pi_\pm = P_{S,\max}|\psi_\pm\rangle\langle\psi_\pm| + (1-P_{S,\max})|\psi_\mp\rangle\langle\psi_\mp|. \tag{15}$$

In the limit of infinitely large program register, $N\to\infty$, the POVM (15) approaches the ideal projective measurement (4).

B. Error-free probabilistic multimeter

The multimeter designed in the preceding section is only approximate, because the effective POVM (15) on the data register differs from the projective measurement in the basis $|\psi_+\rangle$, $|\psi_-\rangle$. Here, we construct a multimeter that realizes an *exact* von Neumann measurement in the basis (1) with some probability P_S. This is achieved at the expense of the inconclusive results which occur with the probability $P_I=1-P_S$ and are associated with the POVM element $\Pi_?$. Such a *probabilistic* multimeter must unambiguously discriminate between two mixed states R_+ and R_-. The unambiguous discrimination of mixed quantum states [13,14] (and, more generally, discrimination of mixed states with inconclusive results [15,16]) has attracted considerable attention recently.

As formally stated in Ref. [13], we have to find a three-component POVM $\Pi_+,\Pi_-,\Pi_?$ that maximizes the success rate (8) under the constraints

$$\mathrm{Tr}[\Pi_+R_-]=\mathrm{Tr}[\Pi_-R_+]=0,$$

$$\Pi_+ + \Pi_- + \Pi_? = \mathbb{1}, \tag{16}$$

$$\Pi_+ \geqslant 0, \quad \Pi_- \geqslant 0, \quad \Pi_? \geqslant 0,$$

which is an instance of the so-called semidefinite program. The first constraint guarantees that the multimeter will never respond with a wrong outcome, i.e., $\Pi_-(\Pi_+)$ cannot be detected when the data register is in the basis state $|\psi_+\rangle(|\psi_-\rangle)$. The second and third constraints express the completeness of the POVM and the positive semidefiniteness of the POVM elements.

Here we shall give a simple intuitive construction of the optimal POVM and we shall analyze the dependence of P_I on N. The optimality of the POVM will be formally proved in the next subsection using the techniques introduced in Ref. [15].

Due to the particular structure of the operators R_+ and R_- the problem of unambiguous discrimination of R_+ and R_- splits into N independent problems of unambiguous discrimination of two *pure* states $|\varphi_{N,k}^+\rangle$ and $|\varphi_{N,k}^-\rangle$. The unambiguous discrimination of two pure nonorthogonal states with equal *a priori* probabilities has been studied by Ivanovic [17], Dieks [18], and Peres [19]. The minimal probability of inconclusive results is equal to the absolute value of the scalar product of the two states. Taking this into account, we can immediately write down P_I for the optimal unambiguous phase-covariant multimeter with program $|\psi_+\rangle^{\otimes N}$,

$$P_I=\frac{1}{2^{N+1}}\sum_{k=1}^{N} C_{N+1,k}|\langle\varphi_{N,k}^+|\varphi_{N,k}^-\rangle| + \frac{1}{2^N}. \tag{17}$$

The contribution 2^{-N} to P_I stems from the term X that is common to both operators $R_\pm$. On inserting the expression (9) into Eq. (17) we obtain

$$P_I=\frac{1}{2^{N+1}}\sum_{k=1}^{N} |C_{N,k}-C_{N,k-1}| + \frac{1}{2^N}. \tag{18}$$

We must distinguish the cases of odd and even N. Let us assume that N is even $(N=2n)$. We divide the sum in Eq. (18) into two parts $k\leqslant N/2$ and $k>N/2$ and we find

$$\sum_{k=1}^{N} |C_{N,k}-C_{N,k-1}| = 2\binom{N}{N/2}-2. \tag{19}$$

On inserting the sum back into Eq. (18) we obtain

$$P_I(2n)=\frac{1}{2^{2n}}\binom{2n}{n}. \tag{20}$$

The calculation for odd $N=2n-1$ proceeds along similar lines, and one obtains

$$P_I(2n-1)=\frac{1}{2^{2n-1}}\binom{2n-1}{n-1}. \tag{21}$$

It holds that $P_I(2n-1)=P_I(2n)$; hence the error-free probabilistic phase-covariant multimeter with a $(2n-1)$-qubit program is exactly as efficient as the multimeter with a $2n$-qubit program. It is worth noting here that similar behavior has been observed in the context of optimal $1\to N$ phase covariant cloning of qubits [20], where it was found that the global fidelities of clones produced by the $1\to 2n$ and $1\to 2n+1$ cloning machines are equal. The asymptotic behavior of the probability of inconclusive results (20) and (21) can be extracted with the help of Stirling's formula $N! \approx \sqrt{2\pi N}N^N e^{-N}$. On inserting this approximation into Eq. (20) we get $P_I(N)\approx 2/\sqrt{2\pi N}$.

The POVM elements that describe the optimal error-free multimeter can be easily written down as the properly weighted convex sum of the POVM elements that describe the optimal unambiguous discrimination of the states $|\varphi_{N,k}^+\rangle$ and $|\varphi_{N,k}^-\rangle$,

$$\Pi_+=\sum_{k=1}^{N} D_{N,k}^{-1}|\varphi_{\perp,N,k}^-\rangle\langle\varphi_{\perp,N,k}^-|,$$
$$\Pi_-=\sum_{k=1}^{N} D_{N,k}^{-1}|\varphi_{\perp,N,k}^+\rangle\langle\varphi_{\perp,N,k}^+|, \tag{22}$$

and $\Pi_?=\mathbb{1}-\Pi_+-\Pi_-$. Here $|\varphi_{\perp,N,k}^\pm\rangle$ denote states orthogonal to $|\varphi_{N,k}^\pm\rangle$, respectively,

$$|\varphi_{\perp,N,k}^\pm\rangle=\sqrt{B_{N,k}}|0\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}} \mp \sqrt{1-B_{N,k}}|1\rangle_{\mathrm{d}}|N,k-1\rangle_{\mathrm{p}}, \tag{23}$$

and

$$D_{N,k}=\frac{2}{N+1}\max(k,N+1-k). \tag{24}$$

The effective three-component POVM on the data register associated with the POVM (22) reads

$$\pi_\pm=(1-P_I)|\psi_\pm\rangle\langle\psi_\pm|, \quad \pi_?=P_I\mathbb{1}. \tag{25}$$

Note that when performing a generalized measurement described by the POVM (25) the statistics of the subensemble of conclusive results would exactly agree with the statistics obtained by a von Neumann projective measurement in basis $|\psi_\pm\rangle$, so the multimeter indeed exactly probabilistically performs the required measurement on the qubit stored in the data register.

C. Multimeter with a fixed fraction of inconclusive results

The deterministic multimeters and the error-free probabilistic multimeters discussed in the preceding subsections can be considered as special limiting cases of a more general class of optimal multimeters that yield an inconclusive result with probability $P_I = \mathrm{Tr}[\Pi_?(R_+ + R_-)/2]$ and give the correct measurement outcome with probability $P_S \leqslant 1 - P_I$ when the data register is prepared in the basis state $|\psi_+(\phi)\rangle$ or $|\psi_-(\phi)\rangle$ with equal *a priori* probability. It is convenient to introduce the relative success rate

$$P_{RS} = \frac{P_S}{1 - P_I}, \tag{26}$$

which gives the fraction of correct outcomes in the subensemble of conclusive results. Note that P_{RS} can also be interpreted as the average fidelity of the probabilistic multimeter. The optimal multimeter should achieve the maximal possible P_S (hence also P_{RS}) for a given fixed probability of inconclusive results P_I. This class of multimeters is described by a three-component POVM similarly to the unambiguous (error-free) multimeter. Such multimeters in fact perform the optimal discrimination of mixed quantum states R_+ and R_- with a fixed fraction of inconclusive results. This general quantum-state discrimination scenario has been recently analyzed in detail in Refs. [15,16], where it was shown that the optimal POVM must satisfy the following set of extremal equations:

$$\left(\lambda - \frac{1}{2}R_\pm\right)\Pi_\pm = 0, \quad (\lambda - aR_?)\Pi_? = 0 \tag{27}$$

and

$$\lambda - \frac{1}{2}R_\pm \geqslant 0, \quad \lambda - aR_? \geqslant 0. \tag{28}$$

Here $R_? = (R_+ + R_-)/2$ and λ and a are Lagrange multipliers that account for the constraints $\Pi_+ + \Pi_- + \Pi_? = \mathbb{1}$ and

$$\mathrm{Tr}[\Pi_? R_?] = P_I. \tag{29}$$

It follows from the structure of the extremal Eqs. (27) and (28) that the problem of optimal discrimination of two mixed states $R_\pm$ with a fraction of inconclusive results P_I is formally equivalent to the maximization of the success rate of the deterministic discrimination of three mixed states R_+, R_-, and $R_?$ with *a priori* probabilities $p_\pm = 1/[2(a+1)]$ and $p_? = a/(a+1)$. Of course, this equivalence straightforwardly extends to discrimination of n mixed states.

In the present case, the key simplification stems from the observation that the operators $R_\pm$ have a common block-diagonal form, which was already explored in construction of the optimal error-free phase-covariant multimeter. Formally, we can write

$$R_\pm = \frac{1}{2^{N+1}} \bigoplus_{k=0}^{N+1} R_{\pm,k}, \tag{30}$$

where

$$R_{\pm,k} = C_{N+1,k} |\varphi_{N,k}^\pm\rangle\langle\varphi_{N,k}^\pm|, \quad k = 1, \ldots, N,$$

$$R_{\pm,0} = |0\rangle_{\mathrm{d}}\langle 0| \otimes |N,0\rangle_{\mathrm{p}}\langle N,0|,$$

$$R_{\pm,N+1} = |1\rangle_{\mathrm{d}}\langle 1| \otimes |N,N\rangle_{\mathrm{p}}\langle N,N|.$$

Accordingly, the total Hilbert space of the data and the program states $\mathcal{H} = \mathcal{H}_{\mathrm{d}} \otimes \mathcal{H}_{\mathrm{p}}$ can be decomposed into a direct sum of orthogonal $\mathcal{H}_k$, $\mathcal{H} = \oplus_{k=0}^{N+1}\mathcal{H}_k$. The Hilbert spaces $\mathcal{H}_k$ are either two dimensional (spanned by $|0\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}}$ and $|1\rangle_{\mathrm{d}}|N,k-1\rangle_{\mathrm{p}}$) or one dimensional (spanned by $|0\rangle_{\mathrm{d}}|N,0\rangle_{\mathrm{p}}$ or $|1\rangle_{\mathrm{d}}|N,N\rangle_{\mathrm{p}}$). The optimal Π_+, Π_-, $\Pi_?$, and λ also have a block-diagonal structure:

$$\Pi_\pm = \bigoplus_{k=0}^{N+1} \Pi_{\pm,k}, \quad \Pi_? = \bigoplus_{k=0}^{N+1} \Pi_{?,k}, \quad \lambda = \bigoplus_{k=0}^{N+1} \lambda_k. \tag{31}$$

The extremal equations (27) and (28) split into $N+2$ equations

$$\left(\lambda_k - \frac{1}{2}R_{\pm,k}\right)\Pi_{\pm,k} = 0, \quad (\lambda_k - aR_{?,k})\Pi_{?,k} = 0, \tag{32}$$

$$\lambda_k - \frac{1}{2}R_{\pm,k} \geqslant 0, \quad \lambda_k - aR_{?,k} \geqslant 0. \tag{33}$$

We thus have to determine the optimal POVM on each subspace $\mathcal{H}_k$ and then merge the solutions according to Eq. (31). Due to the structure of the operators $R_\pm$, the task reduces to the discrimination of two pure nonorthogonal states $|\varphi_{N,k}^\pm\rangle$ with inconclusive results, which was discussed in detail by Chefles and Barnett [21] and also by Zhang *et al.* [22].

Let us first consider the nondegenerate case $k = 1, \ldots, N$. We have to distinguish the cases $C_{N,k} \geqslant C_{N,k-1}$ (i.e., $k \leqslant [N/2]$) and $C_{N,k} < C_{N,k-1}$ ($k > [N/2]$). We will explicitly present the results for $k \leqslant [N/2]$. The formulas for $k > [N/2]$ are similar and can be obtained by simple exchanges $C_{N,k} \leftrightarrow C_{N,k-1}$ and $|0\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}} \leftrightarrow |1\rangle_{\mathrm{d}}|N,k-1\rangle_{\mathrm{p}}$. The optimal POVM on each subspace $\mathcal{H}_k$ can be written as follows:

$$\Pi_{+,k} = \frac{1}{2\sin^2\Phi_k} |\Phi_{N,k}^+\rangle\langle\Phi_{N,k}^+|,$$

$$\Pi_{-,k} = \frac{1}{2\sin^2\Phi_k} |\Phi_{N,k}^-\rangle\langle\Phi_{N,k}^-|, \tag{34}$$

$$\Pi_{?,k} = (1 - \tan^{-2}\Phi_k)|0\rangle_{\mathrm{d}}\langle 0| \otimes |N,k\rangle_{\mathrm{p}}\langle N,k|,$$

where

$$|\Phi_{N,k}^\pm\rangle = \cos\Phi_k|0\rangle_{\mathrm{d}}|N,k\rangle_{\mathrm{p}} \pm \sin\Phi_k|1\rangle_{\mathrm{d}}|N,k-1\rangle_{\mathrm{p}}. \tag{35}$$

The angle Φ_k is a function of the Lagrange multiplier a. This dependence can be determined by substituting the explicit form of the optimal POVM (34) into the extremal equations (32) and solving the resulting system of linear equations for λ_k and a. After some tedious but otherwise straightforward algebra we obtain

$$\tan\Phi_k = \begin{cases} 1, & a < a_{\mathrm{th,k}}, \\ \sqrt{\dfrac{C_{N,k}}{C_{N,k-1}}}(2a-1), & a \geqslant a_{\mathrm{th,k}}, \end{cases} \tag{36}$$

where $a_{\mathrm{th,k}} = \frac{1}{2}(1+\sqrt{C_{N,k-1}/C_{N,k}})$. The probability of inconclusive results $P_{I,k}$ and the probability of a correct guess $P_{S,k}$ when discriminating the states $|\varphi_{N,k}^{\pm}\rangle$ with the POVM (34) are given by

$$P_{I,k} = \frac{C_{N,k}}{C_{N+1,k}}\left(1 - \frac{1}{\tan^2\Phi_k}\right),$$
$$P_{S,k} = \frac{\cos^2(\Phi_k - \Theta_k)}{2\sin^2\Phi_k}, \tag{37}$$

where $\Theta_k = \arctan(\sqrt{C_{N,k-1}/C_{N,k}})$.

The cases $k=0$ and $k=N+1$ require special treatment because the two states to be discriminated are actually identical. Let us consider the case $k=0$. If $a \neq 1/2$ then the optimal POVM can be formally determined from Eqs. (34) and (36) where the limit $C_{N,k-1} \to 0$ must be considered. One finds that $\Pi_{?,0} = 0$ for $a < 1/2$ while $\Pi_{+,0} = \Pi_{-,0} = 0$ and $\Pi_{?,0} = \mathbb{1}_0$ for $a > 1/2$. A sharp transition occurs at $a = 1/2$ where the optimal POVM changes from a projective measurement to a single-component POVM with all measurement outcomes being interpreted as inconclusive results. The transition at $a = 1/2$ can be described by a single parameter $\eta \in [0,1]$ and we can write

$$\Pi_{\pm,0} = \frac{1}{2}(1-\eta)|0\rangle_{\mathrm{d}}\langle 0| \otimes |N,0\rangle_{\mathrm{p}}\langle N,0|,$$

$$\Pi_{?,0} = \eta|0\rangle_{\mathrm{d}}\langle 0| \otimes |N,0\rangle_{\mathrm{p}}\langle N,0|.$$

Consequently, we have $P_{S,0} = 1/2$, $P_{I,0} = 0$ for $a < 1/2$, $P_{S,0} = 0$, $P_{I,0} = 1$ for $a > 1/2$, and a smooth transition $P_{S,0} = (1-\eta)/2$, $P_{I,0} = \eta$ at $a = 1/2$.

The class of the optimal probabilistic phase-covariant multimeters is thus parametrized by two numbers $a \in [0,1]$ and $\eta \in [0,1]$. If we combine all the above derived results we can express the dependence of P_S on a and η as follows:

$$P_S = \frac{1}{2^{N+1}}\sum_{k=1}^{N} C_{N+1,k}P_{S,k} + \frac{1}{2^{N+1}}(P_{S,0} + P_{S,N+1}), \tag{38}$$

and a similar formula holds also for P_I. Rather than plotting the dependence of P_S and P_I on a and η, we directly show in Fig. 2 the dependence of the relative success rate $P_{RS} = P_S/(1-P_I)$ (i.e., the fidelity of the probabilistic multimeter) on the fraction of inconclusive results P_I. We can see that P_{RS} monotonically grows with P_I, and the point of unambiguous probabilistic operation is indicated by $P_{RS} = 1$, when P_I has the value given by Eqs. (20) and (21). Taking into account the symmetry of the POVM (34) with respect to the exchanges $k \to N-k+1$ and $|0\rangle_{\mathrm{d}} \to |1\rangle_{\mathrm{d}}$, it is easy to show that the effective POVM on the data qubit corresponding to the optimal POVM (34) is given by

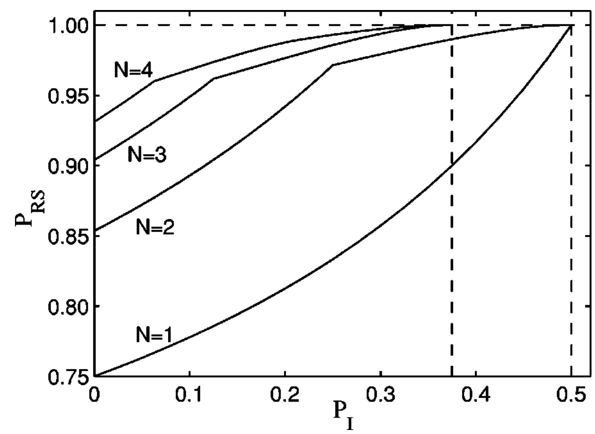

FIG. 2. Dependence of the relative success rate P_{RS} of the optimal phase-covariant multimeter with program $|\psi_+\rangle^{\otimes N}$ on the fraction of inconclusive results P_I.

$$\pi_{\pm} = (1-P_I)[P_{RS}|\psi_{\pm}\rangle\langle\psi_{\pm}| + (1-P_{RS})|\psi_{\mp}\rangle\langle\psi_{\mp}|],$$

$$\pi_? = P_I\mathbb{1}.$$

The POVM has this structure for all possible program states [i.e., all measurement bases (1)] and hence the multimeter is indeed phase covariant. Since the POVM element $\pi_?$ is proportional to the identity operator, the detection of an inconclusive result does not provide any information on the data state.

D. Optical implementation of the simplest phase-covariant multimeter

So far our discussion has been very general and abstract, without referring to any particular physical system. In this section, we propose a feasible optical implementation of the simplest probabilistic phase-covariant multimeter with a single-qubit program. In the suggested scheme, both data and program qubits are encoded as polarization states of single photons, and we thus need to discriminate two nonorthogonal two-photon states. In this context it is worth noting that the unambiguous discrimination of two nonorthogonal polarization states of a single photon has been experimentally demonstrated [23,24], and more involved all-optical schemes for discrimination of coherent states [25] or qudits represented by single photons in several spatial modes [26] have been suggested.

Consider now the phase-covariant multimeter with single-qubit program ($N=1$). In this case, the optimal POVM (34) (acting on data and program qubits together) reads

$$\Pi_{\pm} = |\Psi^{\pm}\rangle\langle\Psi^{\pm}| + \frac{1-\eta}{2}(|0\rangle_{\mathrm{d}}\langle 0| \otimes |0\rangle_{\mathrm{p}}\langle 0| + |1\rangle_{\mathrm{d}}\langle 1| \otimes |1\rangle_{\mathrm{p}}\langle 1|),$$

$$\Pi_? = \eta(|0\rangle_d\langle 0| \otimes |0\rangle_p\langle 0| + |1\rangle_d\langle 1| \otimes |1\rangle_p\langle 1|), \qquad (39)$$

where $|\Psi^{\pm}\rangle=(1/\sqrt{2})(|0\rangle_d|1\rangle_p \pm |1\rangle_d|0\rangle_p)$ and $\eta\in[0,1]$. The relative success rate (the probability of a correct answer in the case of a conclusive result) depends on the probability of an inconclusive result in the following way:

$$P_{RS} = \frac{3-2P_I}{4(1-P_I)}.$$

For $\eta=0$ one has an ambiguous (error-prone) operation with no inconclusive results ($P_I=0, P_{RS}=3/4$) while for $\eta=1$ one gets an unambiguous (error-free) but probabilistic measurement device ($P_I=1/2, P_{RS}=1$).

Clearly, when $\eta=1$ then the POVM (39) is just a projective measurement on the Bell states $|\Psi^+\rangle,|\Psi^-\rangle$ and on the rest of the four-dimensional Hilbert space. If $\eta<1$ we just "reinterpret" some inconclusive results as "conclusive" ones. That is, we will treat randomly selected (with probability $1-\eta$) inconclusive results as results "+" or "−" (at random). Therefore, it is quite enough to consider only the unambiguous version ($\eta=1$) of this phase-covariant quantum multimeter as all the other variants ($0\leq\eta<1$) can be obtained by manipulating the measured data only.

If the states of the data and program qubits are encoded into polarization states of two photons it is possible to distinguish Bell states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ only by means of passive linear optical elements, namely, by a balanced beam splitter and two polarization beam splitters [27–31]. In this way the simplest phase-covariant multimeter could be relatively easily realized experimentally. Such an experiment was just finished in our laboratory in Olomouc and the results will be published elsewhere [32].

E. Optimal multimeter for VMC program

As mentioned in the Introduction, the phase-covariant multimeter can be constructed with the use of the programmable VMC gate [4], which can probabilistically implement any rotation of a qubit about the z axis of the Poincaré sphere, i.e., the transformation

$$U(\phi)|0\rangle = |0\rangle, \quad U(\phi)|1\rangle = e^{i\phi}|1\rangle. \qquad (40)$$

The multimeter measuring in the basis (1) will then consist of a VMC gate programmed to perform the rotation $U(-\phi)$, followed by a measurement in a fixed basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \qquad (41)$$

The VMC gate is probabilistic and it fails with probability $P_I=2^{-N}$ where N is the size of the program register in qubits. The VMC gate requires the following product-state program:

$$|\Psi\rangle_p = \bigotimes_{j=1}^{N} \frac{1}{\sqrt{2}}[|0\rangle + \exp(i2^{j-1}\phi)|1\rangle]. \qquad (42)$$

It is convenient to introduce the notation $|k_{\mathrm{bin},N}\rangle$ which indicates an N-qubit product state where each qubit is in state $|0\rangle$ or $|1\rangle$ and $k_{\mathrm{bin},N}$ stands for the N-bit binary representation for the integer k. To give a few examples, we have $|5_{\mathrm{bin},4}\rangle=|0101\rangle$, $|3_{\mathrm{bin},6}\rangle=|000011\rangle$, etc. With the help of this notation, we can rewrite the program state (42) as follows:

$$|\Psi\rangle_p = \frac{1}{2^{N/2}}\sum_{k=0}^{2^N-1} e^{ik\phi}|k_{\mathrm{bin},N}\rangle_p. \qquad (43)$$

We now determine the optimal phase-covariant multimeter for the N-qubit program (42). All the calculations closely follow those performed in Secs. II A–II C. We therefore omit details and present only the final results. We first need to determine the two operators R_+ and R_-, which can be expressed as

$$R_{\pm} = \frac{1}{2\pi}\int_0^{2\pi} \psi_{\pm}(\phi)\otimes\Psi_p(\phi)d\phi. \qquad (44)$$

On inserting the program (43) into this formula, we obtain

$$R_{\pm} = \frac{1}{2^N}\sum_{k=1}^{2^N-1} |\vartheta^{\pm}_{N,k}\rangle\langle\vartheta^{\pm}_{N,k}| + \frac{1}{2^{N+1}}X, \qquad (45)$$

where X is defined in Eq. (10) and

$$|\vartheta^{\pm}_{N,k}\rangle = \frac{1}{\sqrt{2}}[|0\rangle_d|k_{\mathrm{bin},N}\rangle_p \pm |1\rangle_d|(k-1)_{\mathrm{bin},N}\rangle_p]. \qquad (46)$$

Note, in particular, that the plus and minus states are orthogonal, $\langle\vartheta^{+}_{N,k}|\vartheta^{-}_{N,l}\rangle=0$. This implies that we can rewrite Eq. (45) as

$$R_{\pm} = \frac{1}{2^N}\Pi^V_{\pm,N} + \frac{1}{2^{N+1}}X, \qquad (47)$$

where $\Pi^V_{+,N}$ and $\Pi^V_{-,N}$ are orthogonal projectors, $\mathrm{Tr}[\Pi^V_{+,N}\Pi^V_{-,N}]=0$. The explicit expression for $\Pi^V_{\pm,N}$ is given by the sum on the right-hand side of Eq. (45). In view of Eq. (47), the problem of optimal discrimination between mixed states R_+ and R_- becomes rather trivial. The optimal discrimination strategy is to carry out a measurement described by a three-component POVM

$$\Pi^V_{+,N}, \quad \Pi^V_{-,N}, \quad X. \qquad (48)$$

If we detect the outcome $\Pi^V_{+,N}$ then we are sure that the state $|\psi_+\rangle$ was present in the data register. Similarly, the detection of $\Pi^V_{-,N}$ unambiguously indicates the presence of $|\psi_-\rangle$. Finally, if X is registered, then each basis state could have been present with probability 1/2 and we have no information. So if we want to construct an unambiguous probabilistic multimeter, then we interpret all detections of X as inconclusive results. On the other hand, a deterministic (but error-prone) multimeter is obtained if we interpret the result X randomly with probability 1/2 as outcome $|\psi_+\rangle$ or $|\psi_-\rangle$. The probability of detection of X can easily be determined from Eq. (45) and reads $P_I=2^{-N}$, which is the same as P_I for the VMC programmable gate that can probabilistically implement any rotation (40) [4]. This implies that the optimal phase-covariant multimeter with the program (42) can be realized with the use of the VMC gate described in Ref. [4].

The program (42) is very efficient, because the probability of inconclusive results decreases exponentially with the number of qubits, $P_I = 2^{-N}$. This should be compared with the program $|\psi_+\rangle^{\otimes N}$ considered in Secs. II A–II C. There we showed that for this latter program $P_I \propto 1/\sqrt{N}$ or $(1-P_S) \propto 1/N$ in the case of a deterministic multimeter. However, to make a fair comparison, we should take into account the dimensions of the effective Hilbert spaces that are the supports of the program states. The Hilbert space spanned by the VMC program (43) is the whole Hilbert space of N qubits whose dimension is 2^N. On the other hand, the support of the program $|\psi_+\rangle^{\otimes N}$ is the symmetric subspace $\mathcal{H}_{+,N}$ of the Hilbert space of N qubits and dim $\mathcal{H}_{+,N} = N+1$. Hence, taking into account these Hilbert space dimensions, the scaling of the probabilities of error or inconclusive results obtained for programs (42) and $|\psi_+\rangle^{\otimes N}$ are comparable. Also, for the experimentally interesting case with $N=1$ these two programs are equivalent.

Finally, we would like to briefly comment on the covariance of the programs. By covariance we mean that the program for the phase $\phi+\Delta\phi$ can be obtained from the program corresponding to ϕ by rotating each program qubit by the amount $\Delta\phi$ according to Eq. (40). This covariance property is satisfied by the program $|\psi_+\rangle^{\otimes N}$ but it clearly does not hold for the program (42). In applications where this kind of covariance of the program is required, one should therefore use the multimeter with the program $|\psi_+\rangle^{\otimes N}$.

III. UNIVERSAL MULTIMETERS FOR QUBITS

In this section we will relax the confinement to bases consisting of vectors from the equator of the Bloch sphere and will study universal multimeters designed for measurement in *any* basis represented by two orthogonal states $|\psi_+\rangle = \cos(\vartheta/2)|0\rangle + e^{i\phi}\sin(\vartheta/2)|1\rangle$ and $|\psi_-\rangle = \sin(\vartheta/2)|0\rangle - e^{i\phi}\cos(\vartheta/2)|1\rangle$. We want this measurement basis to be controlled by the quantum state of a program register, $|\Psi(\psi)\rangle_p$. The program will be assumed in the simplest product-state form where both basis states $|\psi_+\rangle$ and $|\psi_-\rangle$ are represented equally, $|\Psi(\psi)\rangle_p = |\psi_+\rangle|\psi_-\rangle$.

A. Deterministic multimeter

First, let us assume the multimeter that always "works" but that allows for some erroneous results. Such a deterministic multimeter was analyzed in Ref. [2]. The optimal (in the sense of the minimum error rate) two-component POVM can be obtained in a similar way as in Sec. II A. In fact, the task is equivalent to the discrimination of two mixed states

$$R_+ = \int_\psi d\psi\, |\Psi_+\rangle\langle\Psi_+|, \qquad R_- = \int_\psi d\psi\, |\Psi_-\rangle\langle\Psi_-|, \tag{49}$$

where the averaging goes over all bases in the qubit space, i.e., over the whole surface of the Bloch sphere, $\int_\psi d\psi = (1/4\pi)\int_0^\pi \int_0^{2\pi} \sin\vartheta\, d\vartheta\, d\phi$, and

$$|\Psi_+\rangle = |\psi_+\rangle_d \otimes |\psi_+\rangle|\psi_-\rangle_p,$$

$$|\Psi_-\rangle = |\psi_-\rangle_d \otimes |\psi_+\rangle|\psi_-\rangle_p.$$

After some algebra we obtain

$$R_\pm = \frac{1}{12}\Pi_{\rm sym} + \frac{1}{3}|A_\pm\rangle\langle A_\pm| + \frac{1}{3}|B_\pm\rangle\langle B_\pm|, \tag{50}$$

where $\Pi_{\rm sym}$ is the projector on the symmetric subspace of three qubits and the eigenvectors $|A_\pm\rangle$ and $|B_\pm\rangle$ can be expressed in the computational basis as follows:

$$\begin{aligned} |A_+\rangle &= \frac{1}{\sqrt{6}}(|0\rangle_d|11\rangle_p + |1\rangle_d|01\rangle_p - 2|1\rangle_d|10\rangle_p), \\ |B_+\rangle &= \frac{1}{\sqrt{6}}(-2|0\rangle_d|01\rangle_p + |0\rangle_d|10\rangle_p + |1\rangle_d|00\rangle_p), \\ |A_-\rangle &= \frac{1}{\sqrt{6}}(-|0\rangle_d|11\rangle_p + 2|1\rangle_d|01\rangle_p - |1\rangle_d|10\rangle_p), \\ |B_-\rangle &= \frac{1}{\sqrt{6}}(-|0\rangle_d|01\rangle_p + 2|0\rangle_d|10\rangle_p - |1\rangle_d|00\rangle_p). \end{aligned} \tag{51}$$

Notice the important orthogonality properties

$$\langle A_+|B_+\rangle = \langle A_-|B_-\rangle = \langle A_+|B_-\rangle = \langle A_-|B_+\rangle = 0,$$

$$\langle A_+|A_-\rangle = \langle B_+|B_-\rangle = \frac{1}{2}.$$

Moreover, the states (51) are also orthogonal to any state from the symmetric subspace of three qubits.

As shown in Ref. [2], the optimal POVM for the deterministic discrimination of the mixed states (50) has the following form:

$$\Pi_+ = \frac{1}{2}\Pi_{\rm sym} + |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|, \qquad \Pi_- = \mathbb{1} - \Pi_+, \tag{52}$$

where $\mathbb{1}$ is an identity operator on the Hilbert space of three qubits, and

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)|0\rangle_d|01\rangle_p - (\sqrt{3}-1)|0\rangle_d|10\rangle_p - 2|1\rangle_d|00\rangle_p], \\ |\phi_2\rangle &= \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)|1\rangle_d|10\rangle_p - (\sqrt{3}-1)|1\rangle_d|01\rangle_p - 2|0\rangle_d|11\rangle_p]. \end{aligned} \tag{53}$$

The corresponding maximal success rate (probability of a correct result) is

$$P_{S,\max} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right).$$

For any program $|\psi_+\rangle|\psi_-\rangle_p$ the effective POVM on the data qubit is given by Eq. (15); hence the multimeter is universal and works equally well for all bases.

B. Probabilistic error-free multimeter

Let us now deal with the situation when we want to avoid any errors. So we are looking for such a three-component POVM $(\Pi_+,\Pi_-,\Pi_?)$ acting on data and program together that gives one of the three results $(+,-,?)$ according to the following prescription:

$$|\psi_+\rangle_d \otimes |\psi_+\rangle|\psi_-\rangle_p \to +\text{ or } ?\,,$$

$$|\psi_-\rangle_d \otimes |\psi_+\rangle|\psi_-\rangle_p \to -\text{ or } ?\,,$$

where ? indicates an inconclusive result. As in Sec. II, the mean probability of an inconclusive result is defined by $P_I = \frac{1}{2}\mathrm{Tr}[\Pi_?(R_+ + R_-)]$ and $\Pi_?$ is the POVM component corresponding to an inconclusive result.

Our aim is to find the POVM that never wrongly identifies states $|\Psi_+\rangle$ and $|\Psi_-\rangle$ for any choice of basis $|\psi_\pm\rangle$ and that, at the same time, minimizes the probability of an inconclusive result. This problem is formally equivalent to the determination of the optimal POVM for unambiguous discrimination of two mixed states R_+ and R_-. It means that, as in Sec. II B, we are looking for operators $\Pi_+,\Pi_-,\Pi_?$ minimizing P_I under the constraints (16), where the relevant $R_\pm$ are defined by Eq. (49).

The optimal POVM for the unambiguous discrimination of these two mixed states consists of multiples of projectors onto the kernels of R_+ and R_- (and of the supplement to unity). The outcome Π_+ can be invoked only by R_+, the outcome Π_- only by R_-. We get

$$\Pi_+ = \frac{2}{3}[|\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2|],$$

$$\Pi_- = \frac{2}{3}[|\kappa_1\rangle\langle\kappa_1| + |\kappa_2\rangle\langle\kappa_2|], \qquad (54)$$

$$\Pi_? = \mathbb{1} - \Pi_+ - \Pi_-,$$

where

$$|\chi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|01\rangle_p - |1\rangle_d|00\rangle_p),$$

$$|\chi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|11\rangle_p - |1\rangle_d|10\rangle_p),$$

$$|\kappa_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|10\rangle_p - |1\rangle_d|00\rangle_p),$$

$$|\kappa_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|11\rangle_p - |1\rangle_d|01\rangle_p).$$

This POVM leads to the lowest probability of an inconclusive result, which equals 2/3.

The proof of optimality follows the same lines as in Sec. II B. Due to the particular structure of operators R_+ and R_- the problem of their unambiguous discrimination splits into independent problems of the unambiguous discrimination of two *pure* states. This can be most easily seen from the spectral decomposition of R_+ and R_- [cf. Eq. (50)]. Each operator $R_\pm$ possesses a two-dimensional kernel and the matrix representations of R_+ and R_- exhibit a common block-diagonal structure. The first block (associated with eigenvalue 1/12) corresponds to the four-dimensional symmetric subspace of three qubits. The second and third blocks (associated with eigenvalue 1/3) correspond to two-dimensional spaces spanned by $\{|A_+\rangle,|A_-\rangle\}$ and $\{|B_+\rangle,|B_-\rangle\}$, respectively. Clearly, our discrimination problem reduces to the unambiguous discrimination of states $|A_+\rangle,|A_-\rangle$, and $|B_+\rangle|B_-\rangle$, respectively.

Thus the minimal overall probability of the inconclusive result is

$$P_I = \frac{1}{3}(|\langle A_+|A_-\rangle| + |\langle B_+|B_-\rangle|) + \frac{4}{12} = \frac{2}{3}.$$

The term 4/12 stems from the totally symmetric states, which are the same for both operators $R_\pm$.

C. Multimeter with a fixed fraction of inconclusive results

Now we relax the requirement of unambiguous (error-free) operation. Thus our task is as follows: For a given probability of an inconclusive result minimize the error rate (i.e., maximize the success rate) or vice versa. We have already seen the two limit cases: the deterministic and the probabilistic error-free multimeters as described above.

The optimal discrimination of two mixed states $R_\pm$ with a fraction of inconclusive results P_I is formally equivalent to the maximization of the success rate of the deterministic discrimination of three mixed states R_+, R_-, and $R_? = (R_+ + R_-)/2$ with *a priori* probabilities $p_\pm = 1/[2(a+1)]$ and $p_? = a/(a+1)$, where $a \in [0,1]$ is a certain Lagrange multiplier [15,16]. Again, we can profitably use the specific structure of operators $R_\pm$ described in the preceding subsection. The method of calculation is the same as in Sec. II C.

Let us start with the discrimination of vectors from the symmetric subspace (let $\{|\xi_i\rangle\}_i$ be an orthonormal basis in $\mathcal{H}_{\text{sym}}$). Because the vectors $|\xi_i\rangle$ are the same for both $R_\pm$ we simply try to discriminate identical states. It was shown in Sec. II C that for $a < 1/2$ the POVM component corresponding to the inconclusive result $\Pi_{?,i} = 0$ and for $a > 1/2$, contrariwise, the conclusive-result components are zero, $\Pi_{\pm,i} = 0$. For the boundary value $a = 1/2$ there is a smooth transition:

$$\Pi_{\pm,i} = \frac{1}{2}(1-\eta)|\xi_i\rangle\langle\xi_i|,$$

$$\Pi_{?,i} = \eta|\xi_i\rangle\langle\xi_i|, \quad \eta \in [0,1].$$

The success rates and inconclusive-result rates are given in Table I.

Now we can proceed to the discrimination (with a given inconclusive-result fraction) of states $|A_+\rangle$ and $|A_-\rangle$ defined

TABLE I. Success rate and probability of inconclusive results as functions of a when discriminating two identical states.

	$a<1/2$	$a=1/2$	$a>1/2$
P_S''	1/2	$(1-\eta)/2$	0
P_I''	0	η	1

by Eqs. (51). For states $|B_+\rangle$ and $|B_-\rangle$ the calculation is completely analogous and the results for success and inconclusive-result rates are the same. States $|A_\pm\rangle$ include the angle 60° and they can be expressed in the following way:

$$|A_\pm\rangle = \frac{1}{2}(\sqrt{3}|\beta\rangle \pm |\alpha\rangle),$$

where

$$|\alpha\rangle = \frac{1}{\sqrt{6}}(2|0\rangle_d|11\rangle_p - |1\rangle_d|01\rangle_p - |1\rangle_d|10\rangle_p),$$

$$|\beta\rangle = \frac{1}{\sqrt{2}}(|1\rangle_d|01\rangle_p - |1\rangle_d|10\rangle_p).$$

The POVM for the optimal discrimination can be written as follows:

$$\Pi_{\pm,A} = \frac{1}{2\sin^2\Phi}|\Xi_\pm\rangle\langle\Xi_\pm|,$$

$$\Pi_{?,A} = \left(1 - \frac{1}{\tan^2\Phi}\right)|\beta\rangle\langle\beta|, \tag{55}$$

where

$$|\Xi_\pm\rangle = \cos\Phi|\beta\rangle \pm \sin\Phi|\alpha\rangle. \tag{56}$$

We can imagine this POVM in the following geometrical way. We start with $\Phi=45°$ so that the states $|\Xi_\pm\rangle$ are orthogonal. This situation corresponds to the Helstrom deterministic (but error-prone) discrimination. Then, increasing Φ, the vectors $|\Xi_\pm\rangle$ move toward each other on the Bloch sphere. Finally, we get to the situation when $|\Xi_+\rangle$ is orthogonal to $|A_-\rangle$ and $|\Xi_-\rangle$ is orthogonal to $|A_+\rangle$; $\Phi=60°$. This case corresponds to the unambiguous discrimination of states $|A_\pm\rangle$.

Now one can easily calculate the probability of success:

$$P_S' = \frac{1}{8}\left(\frac{\sqrt{3}}{\tan\Phi} + 1\right)^2, \tag{57}$$

and the probability of an inconclusive result:

$$P_I' = \frac{3}{4}\left(1 - \frac{1}{\tan^2\Phi}\right). \tag{58}$$

It follows from the extremal equations that

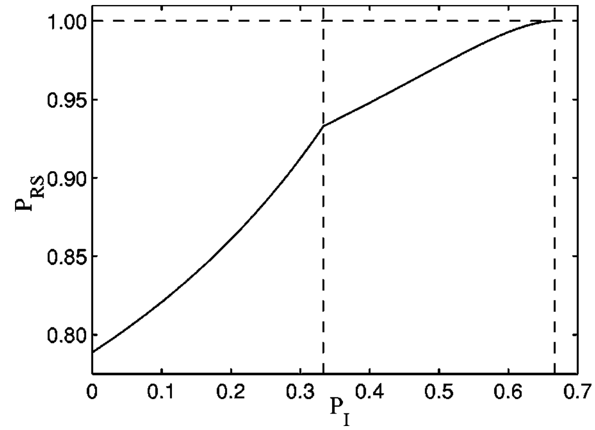

FIG. 3. Dependence of the relative success rate P_{RS} on the probability of inconclusive results P_I, for optimal universal multimeter with program $|\psi_+\rangle|\psi_-\rangle$.

$$\tan\Phi = \begin{cases} 1 & \text{for} \quad a < \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right), \\ \sqrt{3}(2a-1) & \text{for} \quad a \geq \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right). \end{cases}$$

At this stage we are ready to write down the total success rate and inconclusive-result rate for the discrimination of states $R_\pm$. Clearly,

$$P_S = \frac{1}{3}P_S'' + \frac{2}{3}P_S', \quad P_I = \frac{1}{3}P_I'' + \frac{2}{3}P_I'.$$

We can also introduce the relative success rate (i.e., the success rate calculated only for "conclusive" results): $P_{RS} = P_S/(1-P_I)$.

One must examine four different sets of parameter a: $a \in [0,\frac{1}{2})$, $a=\frac{1}{2}$, $a \in (\frac{1}{2},\frac{1}{2}(1+1/\sqrt{3})]$, and $a \in (\frac{1}{2}(1+1/\sqrt{3}),1]$. Finally, it can be seen that (see also Fig. 3)

$$P_S = \begin{cases} \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) - \frac{P_I}{2} & \text{if} \quad 0 \leq P_I \leq \frac{1}{3}, \\ \frac{1}{2} - \frac{P_I}{2} + \frac{1}{3}\sqrt{\frac{5}{4} - \frac{3P_I}{2}} & \text{if} \quad \frac{1}{3} < P_I \leq \frac{2}{3}. \end{cases} \tag{59}$$

For $P_I=2/3$ the error-free operation ($P_{RS}=1$) is approached and there is no reason to increase P_I further.

Apparently, the optimal POVM for universal multimeters with a fixed fraction of inconclusive results has two different forms according to the value of the probability of the inconclusive result. First let us write down the POVM for $P_I \in [0,\frac{1}{3}]$:

$$\Pi_\pm = \Pi_\pm^D - \frac{3}{2}P_I\Pi_{sym},$$

$$\Pi_? = 3P_I\Pi_{sym}, \tag{60}$$

where $\Pi_\pm^D$ denote the elements of the POVM for deterministic discrimination that are defined by Eq. (52).

When $P_I \in \left(\frac{1}{3}, \frac{2}{3}\right]$ the POVM can be expressed as

$$\Pi_\pm = \Pi_{\pm,A} + \Pi_{\pm,B}, \tag{61}$$

$$\Pi_? = \Pi_{\rm sym} + \Pi_{?,A} + \Pi_{?,B},$$

where $\Pi_{\pm,B}$ and $\Pi_{?,B}$ are POVM elements for discrimination of vectors $|B_\pm\rangle$ that can be obtained in a completely analogous way as that for vectors $|A_\pm\rangle$ [see Eq. (55)]. For $P_I=\frac{1}{3}$ the two POVMs (60) and (61) coincide [notice that for $\Phi=45°$, $|\Xi_+\rangle=-|\phi_2\rangle$ as follows from Eqs. (53) and (56)].

The operation of the multimeter for different values of P_I can be figured as follows: When P_I grows from zero it is most advantageous to gradually move the contributions that are the same for both $R_\pm$ and that substantially contribute to errors from conclusive to inconclusive results. This means that the multiple of the projector to the symmetric subspace increases in $\Pi_?$. When $P_I=1/3$ then $\Pi_?=\Pi_{\rm sym}$, and further increase of the fraction of $\Pi_{\rm sym}$ is impossible (because $\Pi_\pm$ and $\Pi_?$ must form a POVM). If one wants to increase P_I further above $1/3$ he/she must start to turn the vectors $|\Xi_\pm\rangle$ as described above. The point $P_I=2/3$ corresponds to the unambiguous discrimination.

IV. UNIVERSAL PROBABILISTIC ERROR-FREE MULTIMETER FOR QUDITS

Let us consider a multimeter that could realize an arbitrary von Neumann projective measurement on a single d-level system (qudit). Let $|\psi_j\rangle$, $j=1,\ldots,d$ denote orthonormal-basis states. We consider the conceptually simplest program that consists of the d qudits in basis states,

$$|\Psi\rangle_{\rm p} = |\psi_1\rangle|\psi_2\rangle\cdots|\psi_j\rangle\cdots|\psi_d\rangle \equiv [U_d(g)]^{\otimes d}|\Psi_0\rangle_{\rm p}, \tag{62}$$

where $|\Psi_0\rangle_{\rm p}=|1\rangle|2\rangle\cdots|j\rangle\cdots|d\rangle$, $U_d(g)$ is a unitary operation acting on the basis states according to $U_d(g)|j\rangle=|\psi_j\rangle$, and $g\in {\rm SU}(d)$. We are interested in the probabilistic error-free multimeter which can respond with an inconclusive outcome but never makes an error, i.e., $\pi_j \propto |\psi_j\rangle\langle\psi_j|$. The multimeter is described by a $(d+1)$-component POVM on $d+1$ qudits (the data qudit and d program qudits). The POVM $\{\Pi_1,\ldots,\Pi_d,\Pi_?\}$ should optimally unambiguously discriminate among d mixed states

$$R_j = \int_{SU(d)} U_d(g)|j\rangle_{\rm data}\langle j|U_d^\dagger(g)\otimes[U_d(g)]^{\otimes d}|\Psi_0\rangle_{\rm p}\langle\Psi_0| \times[U_d^\dagger(g)]^{\otimes d}d\mu(g), \tag{63}$$

where the integration is carried over the whole group SU(d) with the invariant Haar measure $d\mu(g)$.

We conjecture that the optimal POVM elements Π_j have the following structure:

$$\Pi_j = C|\Sigma_d^-\rangle_{\bar{j}}\langle\Sigma_d^-| \otimes \mathbb{1}_j, \tag{64}$$

$$\Pi_? = \mathbb{1} - \sum_{j=1}^{d}\Pi_j,$$

where $|\Sigma_d^-\rangle_{\bar{j}}$ is the totally antisymmetric state of d qudits: the data qudit and all program qudits except for the jth qudit, and $\mathbb{1}_j$ stands for the identity operator on the Hilbert space of the jth program qudit. We can write

$$|\Sigma_d^-\rangle_{\bar{j}} = \frac{1}{\sqrt{d!}}\sum_i \epsilon_i|i_1\rangle_{\rm data}\otimes|i_2,\ldots,i_d\rangle_{{\rm p}\bar{j}}, \tag{65}$$

where we sum over all permutations of $\{1,2,\ldots,d\}$ and ϵ_i is the sign of the permutation. Clearly, vectors $|\Sigma_d^-\rangle_{\bar{j}}\otimes|x\rangle_j$, where $|x\rangle_j$ is an arbitrary state of the jth program qudit, are orthogonal to any vector $|\Psi_k\rangle=|\psi_k\rangle_{\rm data}|\psi_1\rangle|\psi_2\rangle\cdots|\psi_d\rangle$ with $k\neq j$. It is easy to verify that ${\rm Tr}[\Pi_j R_k]\propto\delta_{jk}$. This means that the only contribution to the outcome Π_j can originate from the jth basis state of the data qudit.

Clearly, the POVM (64) is a POVM describing a probabilistic unambiguous multimeter. We believe it is even the optimal one for the program (62). This hypothesis is based on the conjecture that the kernels of operators R_j have the form $\mathcal{K}_j=\mathcal{H}_j^{\rm ant}\otimes\mathbb{1}_{\bar{j}}$ where $\mathcal{H}_j^{\rm ant}$ is the antisymmetric space of two qudits—the data one and the jth program one. The symbol $\mathbb{1}_{\bar{j}}$ denotes the identity operator on $d-1$ program qudits exclusive of the jth qudit. (At worst, $\mathcal{K}_j$ are the subspaces of the appropriate kernels.) The d-dimensional subspace spanned by $|\Sigma_d^-\rangle_{\bar{j}}\otimes|x\rangle_j$, where $|x\rangle_j$ is an arbitrary state of the jth qudit, represents an intersection of $d-1$ spaces $\mathcal{K}_k$ (excluding the jth one): $\cap_{k=1, k\neq j}^{d}\mathcal{K}_k$.

The sum of the d POVM elements Π_j must be lower than the identity operator, $\sum_{j=1}^d \Pi_j \leq \mathbb{1}$, which imposes a constraint on the normalization factor C. Since we want to maximize the probability of success we must choose the maximum possible C, which can be expressed in terms of the maximum eigenvalue of the operator

$$Y = \sum_{j=1}^{d}|\Sigma_d^-\rangle_{\bar{j}}\langle\Sigma_d^-| \otimes \mathbb{1}_j.$$

The maximal admissible C reads

$$C = \{\max[{\rm eig}(Y)]\}^{-1}. \tag{66}$$

Instead of looking for the maximum eigenvalue of Y we can equivalently calculate the maximum eigenvalue of the operator

$$Z = \sum_{j=1}^{d}|f_j\rangle\langle f_j|, \tag{67}$$

where $|f_j\rangle=|\Sigma_d^-\rangle_{\bar{j}}|1\rangle_j$. The d linearly independent states $|f_j\rangle$ span a d-dimensional Hilbert space $\mathcal{H}_f$. We can write $|f_j\rangle=M|e_j\rangle$ where $|e_j\rangle$ form an orthonormal basis in $\mathcal{H}_f$. On inserting this expression into Eq. (67) we find that

$$Z=\sum_{j=1}^{d} M|e_j\rangle\langle e_j|M^\dagger = MM^\dagger, \tag{68}$$

where the completeness of the basis $|e_j\rangle$ on $\mathcal{H}_f$ has been used. It holds for any square matrix M that $MM^\dagger$ has the same eigenvalues as $F=M^\dagger M$. In the basis $|e_j\rangle$ the matrix elements of F read $F_{jk}=\langle f_j|f_k\rangle$. We thus have to determine the scalar products of the nonorthogonal states $|f_j\rangle$. Let us introduce unnormalized states of $d-1$ qudits, $|\sigma^-_{d-1}\rangle_{\overline{jk}}$, that are obtained by projecting the kth program qudit of the state $|\Sigma^-_d\rangle_{\bar{j}}$ onto state $|1\rangle_k$. It follows that F_{jk} can be calculated as a scalar product of $|\sigma^-_{d-1}\rangle_{\overline{jk}}$ and $|\sigma^-_{d-1}\rangle_{\overline{kj}}$,

$$F_{jk}={}_{\overline{jk}}\langle\sigma^-_{d-1}|\sigma^-_{d-1}\rangle_{\overline{kj}}. \tag{69}$$

It is easy to deduce from the Slater determinant representation of the totally antisymmetric state (65) that $|\sigma^-_{d-1}\rangle_{\overline{jk}}$ is also a totally antisymmetric state of the data qudit and all the program qudits except the jth and kth ones,

$$|\sigma^-_{d-1}\rangle_{\overline{jk}}=\frac{(-1)^t}{\sqrt{d!}}\sum_i \epsilon'_i|i_1\rangle_{\text{data}}\otimes|i_2,\ldots,i_{d-1}\rangle_{p,\overline{jk}}, \tag{70}$$

where one sums over all permutations of $\{2,3,\cdots,d\}$, ϵ'_i is the sign of the permutation, and

$$t=\begin{cases} k & \text{for} \quad j>k,\\ k-1 & \text{for} \quad j<k. \end{cases}$$

Assuming $j\neq k$ and inserting the expressions (70) into Eq. (69), we immediately find that

$$F_{jk}=\frac{(d-1)!}{d!}(-1)^{j+k-1}, \quad j\neq k. \tag{71}$$

Since $|f_j\rangle$ are normalized we finally have

$$F_{jk}=\delta_{jk}+(1-\delta_{jk})\frac{(-1)^{j+k-1}}{d}. \tag{72}$$

The operator F can be easily diagonalized,

$$F=\left(1+\frac{1}{d}\right)\mathbb{1}-|\varphi_d\rangle\langle\varphi_d|, \tag{73}$$

where $|\varphi_d\rangle=(1/\sqrt{d})\Sigma_{j=1}^{d}(-1)^j|e_j\rangle$. It follows from Eq. (73) that the largest eigenvalue of F is $\mu_{\max}=1+1/d$; hence $C=d/(d+1)$ and the normalized POVM (64) reads

$$\Pi_j=\frac{d}{d+1}|\Sigma^-_d\rangle_{\bar{j}}\langle\Sigma^-_d|\otimes\mathbb{1}_j. \tag{74}$$

By construction, the probabilistic multimeter is universal and the probability of success

$$P_S=\frac{d}{(d+1)!} \tag{75}$$

does not depend on the particular basis chosen by the program state or on the basis state $|\psi_j\rangle$ sent to the data register. Consequently, the multimeter indeed probabilistically implements the projective measurement in the basis $\{|\psi_j\rangle\}_{j=1}^{d}$, and the effective POVM on the data qudit reads

$$\pi_j=P_S|\psi_j\rangle\langle\psi_j|, \quad j=1,\ldots,d,$$
$$\pi_?=(1-P_S)\mathbb{1}. \tag{76}$$

V. CONCLUSIONS

In this paper we have investigated a broad class of quantum multimeters that can perform a projective measurement on a single data qubit (or qudit). The main feature of the quantum multimeters is that the measurement basis is controlled by the quantum state of the program register, which serves as a kind of quantum "software," while the multimeter itself (quantum "hardware") performs a fixed joint measurement on the data and program registers.

In our investigations we have assumed a finite-dimensional program register, consisting of several qubits (or qudits). In this case it is impossible to design the perfect multimeter that would perform exactly and deterministically a projective measurement in any basis from a continuous set, with the basis being determined by the state of the program register. The multimeters designed here are therefore only approximate. Two conceptually different approximations have been considered. In the first case, the multimeter operates deterministically and always produces an outcome, but the effective measurement on the data deviates from the ideal projective measurement. Such errors are avoided in the second approach, when the multimeter is a probabilistic device whose operation sometimes fails but, when it succeeds, carries out exactly the desired projective measurement.

We have demonstrated that these two kinds of multimeters are in fact just limit cases from a whole class of probabilistic multimeters that are characterized by a certain fraction P_I of inconclusive results. For a fixed dependence of the program on the measurement basis, the problem of designing the optimal multimeter is formally equivalent to finding the optimal POVM for discrimination of mixed states. With the help of the recently developed theory of optimal probabilistic discrimination of mixed quantum states, we have been able to determine analytically the optimal phase-covariant multimeter for an N-qubit program $|\psi_+\rangle^{\otimes N}$ as well as a universal multimeter with a two-qubit program $|\psi_+\rangle|\psi_-\rangle$. Remarkably, in both cases the success rate of the optimal deterministic multimeter exactly coincides with the optimal fidelity of estimation of the basis state $|\psi_+\rangle$ from a single copy of the program state.

We have also proposed a generalization of the probabilistic error-free multimeter to qudits, assuming that the d-qudit program consists of a product of the d basis states. The construction of this multimeter is inspired by the structure of the optimal probabilistic multimeter for qubits and relies on projections on a totally antisymmetric state of d qudits.

Our findings clearly illustrate that the measurement on the data qubit can be quite efficiently controlled by the quantum state of the program register. In particular, we emphasize that a classical description of the measurement basis would require infinitely many bits of classical information, while only a few quantum bits suffice in the present case to obtain an *error-free* (although probabilistic) operation. Our results also reveal many intriguing connections between the concept of quantum multimeters, discrimination of quantum states, and optimal quantum-state estimation. This suggests that there might also be links to the related problems of transmitting information about the direction in space [33–35] or about the reference frame [36,37] using quantum states. The main connection between these protocols and programmable quantum multimeters is that in both cases we want to encode into a quantum state some information about a reference frame. In the case of quantum multimeters it is a reference frame in the abstract Hilbert space of quantum states—the measurement basis. It was shown in Refs. [33–37] that the use of entangled states can improve the fidelity of transmission of information. The natural question arises whether by using entangled states as programs one could achieve a higher success rate (for a fixed size of the program register) than with the product-state programs considered in the present paper. More generally, one would ultimately like to know what is the *optimal program* leading to the maximal achievable success rate. This is a highly nontrivial open problem that certainly deserves further investigation.

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