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Discrimination of the Bell states of qudits by means of linear optics

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Abstract

The question of the discrimination of the Bell states of two qudits (i.e., d-dimensional quantum systems) by means of passive linear optical elements and conditional measurements is discussed. A qudit is supposed to be represented by d optical modes containing exactly one photon altogether. From recent results of Calsamiglia it follows that there is no way how to distinguish the Bell states of two qudits for $d > 2$—not even with the probability of success lower than one—without any auxiliary photons in ancillary modes. Following the results of Carollo and Palma it is proved that it is impossible to distinguish even only one such Bell state with certainty (i.e., with the probability of success equal to one), irrespective of how many auxiliary photons are involved. However, it is shown that auxiliary photons can help to discriminate the Bell states of qudits with the high probability of success: A Bell-state analyzer based on the idea of linear optics quantum computation that can achieve the probability of success arbitrarily close to one is described. It requires many auxiliary photons that must be first "combined" into entangled states. © 2001 Published by Elsevier Science B.V.

1. Introduction

Quantum optics and particularly linear optical elements represent important tools for the experimental investigation of the basic features of quantum information transfer and processing and fundamentals of quantum theory. The special in-terest is devoted to the study of entangled states that are not only highly interesting by themselves but that find also the use in quantum teleportation, quantum dense coding, quantum cryptography, quantum computing, etc. Closely related to many of these issues is the so-called Bell-state analysis, i.e., the discrimination of maximally entangled states completing the non-local orthonormal basis of two- or multi-particle quantum system.

The relative success concerning the discrimination of two of the four Bell states of two qudits with certainty by passive linear optical elements [1,2] challenges the question whether it is possible

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to do something similar also for the Bell states of two qudits if their dimension $d > 2$. At the first glance the direct generalization of the approach given in Ref. [2] indicates that the limitations on the probability of success could be less restrictive for $d > 2$ than for $d = 2$. But when one starts to play with possible extensions of the original Innsbruck scheme [1] for qudits he quickly gets into troubles. The recent results of Calsamiglia [3] indicate that it is impossible without additional auxiliary photons. This was the motivation to subject the problem of "generalized" linear optics Bell-state measurement to more detailed analysis.

In the considered scheme, qudits are represented by d modes of radiation (with the same frequencies and polarizations). The total number of photons in these d modes is required to be one. The ith "logical" (or computational) basis state corresponds to the situation when exactly one photon is present in the ith mode. Such an implementation allows us to realize any unitary operation on a single qudit (up to a global phase) in a deterministic way by the means of passive linear optical elements [4].

In fact, we restrict our tools to beam splitters, phase shifters and delay lines. All of them may be electronically switched—the conditionally dynamics is allowed, i.e., some operations may depend on the result of the measurement on selected modes outgoing the "previous" operation. We consider ideal detectors that can distinguish the number of impinging photons (in a given mode). Even if such detectors are not available in practice yet our choice is justified as we seek for fundamental limitations of linear optical devices.

By the Bell states of two qudits we mean the maximally entangled states of the following form (see, e.g., Ref. [5])

$$|\psi_{mn}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left[2\pi i \frac{jn}{d}\right] a_j^\dagger b_{j\oplus m}^\dagger |\text{vac}\rangle, \qquad (1)$$

where $a_k^\dagger$ and $b_k^\dagger$ are bosonic creation operators in corresponding modes of the first and the second qudit, respectively, $|\text{vac}\rangle$ denotes a vacuum state, and $j \oplus m = (j+m) \bmod d$; m and n go from 0 to $d-1$.

2. No-go theorem for the case without auxiliary photons

Calsamiglia has shown [3] that any linear optical device that does not use auxiliary photons in ancillary modes (however, that may include conditional dynamics) cannot unambiguously discriminate any state with Schmidt rank *higher than two* from any set of two-qudit states spanning the whole two-qudit Hilbert space (qudits are supposed to be represented by one-photon states over d modes). This is true even if the probability of successful discrimination is allowed to be lower than one. If there is a non-zero probability of some detection event for some state from the set that has Schmidt rank higher than two then there is always at least one another state from the set that gives a non-zero probability for this detection event too.

It directly follows that there is no way how to distinguish any one of d^2 Bell states of two qudits (for $d > 2$) without error by means of linear optics if no auxiliary photons are involved.

3. Impossibility to discriminate a Bell state with certainty

In the paper of Carollo and Palma [6] it is shown under the same conditions as assumed here (i.e., linear elements, arbitrary number of auxiliary modes, conditional measurements, and photon number detectors are assumed) that any two L-photon states over M modes, randomly chosen from a known set of K states, are completely (i.e., with certainty) distinguishable in the presence of auxiliary photons only if they are completely distinguishable in the absence of auxiliary photons. It also means that if some two states are not completely distinguishable in the case when no auxiliary photons are involved, then they cannot be distinguished even if any finite number of auxiliary photons are employed.

As stated earlier it is not possible in any way to distinguish any one of d^2 Bell states from all of the remaining states by linear optical device with no additional photons if $d > 2$. In other words, to each Bell state there is at least one another Bell state such that these two states cannot be distin-

guished from each other (providing $d > 2$). According to the statement given in the previous paragraph this must stay valid even if auxiliary photons are allowed. So it is impossible to discriminate even one of d^2 Bell states with certainty (with 100% probability of success) irrespective whether auxiliary photons are allowed or not.

4. Efficient Bell-state analyzer with linear optics

Now we will show how to distinguish all the Bell states of two qudits with the probability of success arbitrarily close to one using only linear optical elements (and auxiliary photons).

Recently, a scheme for "non-deterministic" quantum computation with linear optics were proposed by Knill et al. [7,8]. It is based on a non-linear phase shift:

$$\alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle \rightarrow \alpha_0|0\rangle + \alpha_1|1\rangle - \alpha_2|2\rangle,$$

where kets represent number states in a given mode. We will denote this operation "NS". A simple device was designed that can perform this operation with probability $1/4$. It consists of a couple of beam splitters and phase shifters. The effective non-linearity is provided by a measurement process. Two ancillary modes, with a single photon in one of them, and two photon-number detectors are necessary. The successful operations turns up when the first detector registers one and the second detector no photon.

Using two beam splitters and two operation NS one can built a conditional sign-flip gate (C-SIGN) as shown in Fig. 1. This gate inverts the sign of the state vector of two modes when there are exactly one photon in each of them. In the other cases (with at most one photon altogether) the operation does nothing:
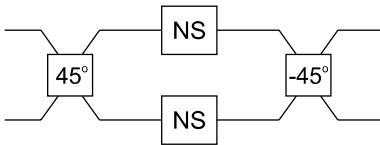

Fig. 1. C-SIGN gate consisting of two beam splitters and two non-linear phase shifters NS.

$$|1\rangle|1\rangle \rightarrow -|1\rangle|1\rangle, \quad |1\rangle|0\rangle \rightarrow |1\rangle|0\rangle,$$
$$|0\rangle|1\rangle \rightarrow |0\rangle|1\rangle, \quad |0\rangle|0\rangle \rightarrow |0\rangle|0\rangle.$$

The unitary matrix representing the transformation of creation operators on the beam splitter is supposed to be

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Particular angles θ are written inside the corresponding boxes in Fig. 1. Since this gate uses two NS operations its probability of success is $1/16$.

However, in the mentioned papers the way is proposed how to increase the probability of the successful action of the C-SIGN arbitrarily close to one. This way is based on a "teleportation trick". It requires preparation of relatively complicated ancillary entangled state of $2n$ photons in $4n$ modes:

$$|\phi_n\rangle = \sum_{j,k=0}^{n} (-1)^{(n-j)(n-k)}|1\rangle^j|0\rangle^{n-j}|0\rangle^j|1\rangle^{n-j}|1\rangle^k$$
$$\times |0\rangle^{n-k}|0\rangle^k|1\rangle^{n-k}, \tag{2}$$

where $|x\rangle^y = |x\rangle|x\rangle \cdots |x\rangle$, y-times. First n modes are "put together" with one input mode and all these $n + 1$ modes are subjected to $n + 1$ point Fourier transform. Then the photon number measurement is performed on the transformed modes and according to the result one of the other n modes is chosen as an output of the gate and its phase is modified in general. The same action is done with the next $2n$ modes and the other input mode; see Fig. 2. It can be shown that the total probability of the success of such a C-SIGN gate is

$$p = \left(\frac{n}{n+1}\right)^2. \tag{3}$$

The Fourier transform, selection of modes, and phase shifts can be implemented by linear optical elements in a deterministic way. The state (2) can be prepared by means of NS operations, beam splitters and phase shifters. The probability of successful preparation can be rather low. But we should stress that this concerns just the preparation of an ancilla. In principle, it can be being prepared in advance and one can try many times. For more details, including the estimation of the success probability of the preparation procedure
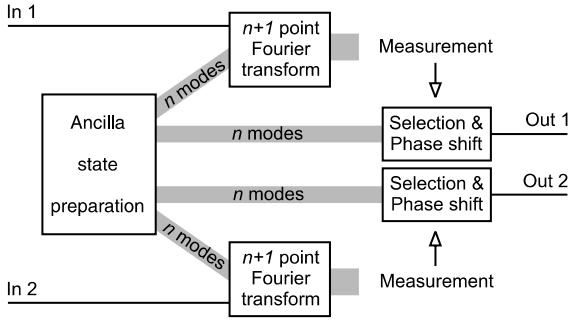
Fig. 2. Schematic view of the setup for C-SIGN gate with the probability of success $p = [n/(n+1)]^2$. Here $4n$ ancillary modes are in the state given by Eq. (2).

and the number of necessary elements, see Refs. [7,8].

Having a C-SIGN gate one can realize a gate "C-SWAP" that conditionally swaps the two modes of radiation (provided there is at most one photon altogether in them). Its scheme is in Fig. 3. If there is one photon in the mode 1 the modes 2 and 3 are swapped. If no photon is present in the mode 1 they are not changed. The probability of the success of the C-SWAP is the same as for the C-SIGN. The gate C-SWAP is the key ingredient to construct a logical operation acting on two qudits x, y of arbitrary dimension d that we will call "C-SHIFT":

$$x \rightarrow x, \quad y \rightarrow (y - x)\mathrm{mod}\,d.$$

For any x this operation represents a cyclic permutation or "rotation" of the values of the second qudit. In total, $d - 1$ rotations are necessary—each of them corresponds to one possible value of the first qudit except the value zero that leads to identity transformation. Any such rotation can be implemented by at most $d - 1$ "transpositions", i.e., C-SWAPs. Thus if the probability of success of
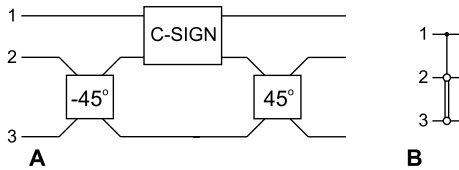

Fig. 3. C-SWAP gate that swaps the modes 2 and 3 if there is a photon in the mode 1. (A) The scheme of the gate built from two beam splitters and one C-SIGN gate. (B) The notation we will use.
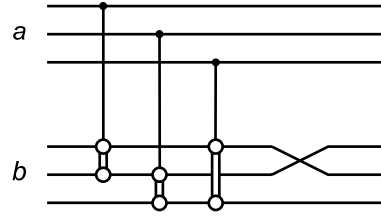

Fig. 4. C-SHIFT gate. An example of the network for $d = 3$ built-up from C-SWAP gates. Here a denotes the control qudit, b the controlled one.

the C-SWAP is p then the total probability of the success of the C-SHIFT is at least $p^{(d-1)^2}$. In particular cases it can be even better, e.g., for $d = 3$ the probability of the success of the C-SHIFT is p^3. Corresponding "network" is in Fig. 4. If one uses C-SIGN gates described above with the probability of success given by Eq. (3) then the probability of the successful action of the C-SHIFT gate is

$$P \geqslant \left(\frac{n}{n+1} \right)^{2(d-1)^2}. \tag{4}$$

Now we can build a Bell-state analyzer. To do it we need our C-SHIFT gate,

$$\text{C-SHIFT}: \quad |x\rangle_1|y\rangle_2 \rightarrow |x\rangle_1|(y - x)\mathrm{mod}\,d\rangle_2, \tag{5}$$

followed by a generalized Hadamard transform acting on the first qudit,

$$\text{HAD}: \quad |x\rangle_1 \rightarrow \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \exp\left[-2\pi \mathrm{i} \frac{kx}{d} \right] |k\rangle_1, \tag{6}$$

where $|j\rangle_l$ represents the jth "logical" state of the lth qudit. The complete setup is shown in Fig. 5. If the sequence of these two operations is applied on a Bell state

$$|\psi_{mn}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left[2\pi \mathrm{i} \frac{jn}{d} \right] |j\rangle_1|(j + m)\mathrm{mod}\,d\rangle_2 \tag{7}$$

[see also Eq. (1)] then the output state of the two qudits is $|n\rangle_1|m\rangle_2$. Thus, if both the qudits are realized as described above the input Bell state can be determined by a simple photodetection.

The generalized Hadamard transform is the unitary transformation of one qudit. As mentioned earlier any such operation can be realized in
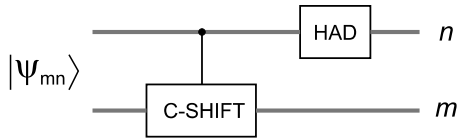
Fig. 5. Scheme of the Bell-state analyzer for two qudits. It consists of a C-SHIFT operation and a generalized Hadamard transform.

a deterministic way with passive linear optical elements (for our implementation of qudits)—simply by combination of beam splitters and phase shifters. The C-SHIFT gate described above is a non-deterministic gate. Its probability of successful action is given by Eq. (4). Clearly, that value determines also the probability of the successful discrimination of an unknown Bell state. Increasing n this probability can be made arbitrarily close to one provided the dimension d is fixed.

Let us note, that the described approach can be extended to the discrimination of "generalized Bell states" of N qudits, i.e., the following maximally entangled states that complete an orthonormal basis in the Hilbert space of N qudits:

$$|\psi_{k_1,k_2,\ldots,k_N}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp\left[2\pi\mathrm{i}\frac{jk_1}{d}\right]$$
$$\times |j\rangle_1 \bigotimes_{i=2}^{N} |(j+k_i)\,\mathrm{mod}\,d\rangle_i, \qquad (8)$$

where $k_i = 0, \ldots, d-1$. In this case the C-SHIFT operation must be applied $N-1$ times: between the first and the Nth qudit, the first and the $(N-1)$th qudit, etc. Finally between the first and the second qudit. Then the generalized Hadamard transform must be performed on the first qudit. If there was a generalized Bell state (8) in the input then the output state reads

$$|k_1\rangle_1 |k_2\rangle_2 \cdots |k_N\rangle_N.$$

5. Conclusions

We have interpreted some recent results concerning unambiguous state discrimination with linear optical elements from the point of view of Bell-state measurement in case of two qudits with

dimension higher than two. This analysis leads to the conclusions that with no auxiliary photons it is impossible to discriminate such Bell states without errors and that it is impossible to discriminate such Bell states with certainty in any way by the means of linear optics.

On the other hand, we have shown by an explicit construction that it is possible, in principle, to build a linear optical Bell-state analyzer capable to discriminate all the states of the Bell basis of two (or even more) qudits with the probability of success arbitrarily close to one. This device is based on the generalization of the idea of linear optics quantum computation for qudits. The price for the high success probability is a complicated setup and a large number of required auxiliary photons in rather complex entangled states.

It seems that the key ingredient that is necessary for the increase of the probability of successful discrimination is the entanglement "added" through the ancilla. The methods of "linear optics quantum computation" enable us to prepare required entangled states from "separated" photons, in principle. But the probability of the success of such a preparation is very low. Besides this, the preparation of single photons represents itself a serious experimental problem as single-photon states are highly non-classical ones. By the way, recently it was shown for all pure input states and for the large class of mixed states that the beam splitter can serve for the preparation of an entangled state on its output only if the input state exhibits non-classical behavior [9].

References

[1] H. Weinfurter, Europhys. Lett. 25 (1994) 559.
[2] J. Calsamiglia, N. Lütkenhaus, Appl. Phys. B 72 (2001) 67, e-print: quant-ph/0007058 (2001).

[3] J. Calsamiglia, e-print: quant-ph/0108108 (2001).

[4] M. Reck, A. Zeilinger, H.J. Bernstein, P. Bertani, Phys. Rev. Lett. 73 (1994) 58.

[5] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W. Wootters, Phys. Rev. Lett. 70 (1993) 1895.

[6] A. Carollo, G.M. Palma, e-print: quant-ph/0106041 (2001).

[7] E. Knill, R. Laflamme, G.J. Milburn, Nature 409 (2001) 46.

[8] E. Knill, R. Laflamme, G.J. Milburn, e-print: quant-ph/0006088 (2000).

[9] M.S. Kim, W. Son, V. Bužek, P.L. Knight, e-print: quant-ph/0106136 (2001).