PROGRAMMABLE QUANTUM MULTIMETERS AND THEIR OPTICAL IMPLEMENTATIONS

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Programmable quantum multimeters are devices that can realize any generalized quantum measurement from a chosen set (either exactly or approximately). Their main feature is that the desired positive operator valued measure (POVM) is selected by the quantum state of a program register. In particular, programmable quantum devices that can accomplish any projective measurement on a single qubit are described. There are two limit cases: Deterministic devices giving an erroneous result from time to time and probabilistic devices that operate error-free but sometimes lead to an inconclusive result. The intermediate cases are also discussed as well as the optimality of such multimeters. Another example of a quantum multimeter is a programmable device for unambiguous discrimination of pairs of non-orthogonal states. Simple optical implementations of some of these devices are proposed and experimental results are presented.

1   Introduction

By quantum multimeters we mean programmable quantum measurement devices that can accomplish any desired POVM from a chosen set. Particular POVM on the "data register" is selected by the quantum state of a "program register".[1] Even non-orthogonal states of the program register can correspond to different POVM's. The multimeter itself is represented by a *fixed* joint POVM on the data and program systems together. Each outcome is associated with the corresponding outcome of the "programmed" POVM on the data alone. From the mathematical point of view the realization of a particular quantum multimeter is equivalent to finding POVM for the discrimination of certain mixed states.

In this contribution we will deal with several examples of quantum multimeters including two experimental implementations. First, we will focus on the *projective measurements on a qubit*. I.e., programmable quantum multimeters that can accomplish any von Neumann measurement on a single qubit will be considered. Because it is impossible to encode an arbitrary projective measurement on a qubit into a finite-dimensional program state exactly they

can do it only approximately.[1,2] However, there are still two different ways how to "approximate" such measurements. The first one can be called "deterministic". Such multimeters always give a result but errors may appear. The second way is "probabilistic". These multimeters fail from time to time (i.e., give an inconclusive result with a certain probability) but in the case of successful operation they never make an error. Second, we will discuss an *adjustable discrimination of two non-orthogonal states*. Programmable quantum device for error-free discrimination of various pairs of non-orthogonal states of a qubit will be briefly described. The selection of the pair of states to be discriminated unambiguously depends on the state of a program register.

2 Projective measurements on a qubit

Let us suppose we would like to measure a qubit in the basis represented by two orthogonal vectors $|\psi\rangle$ and $|\psi_\perp\rangle$. We want this measurement basis be controlled by the quantum state of a program register, $|\phi_p(\psi)\rangle$. An *ideal* multimeter should give two results, 0 and 1, according to

$$|\psi\rangle \otimes |\phi_p(\psi)\rangle \to 0, \qquad |\psi_\perp\rangle \otimes |\phi_p(\psi)\rangle \to 1.$$

Such an ideal measurement cannot be implemented exactly for all ψ's. So, our task is to find such a POVM (on the data qubit and the program register) that represents the *closest approximation* to this demand.

2.1 Deterministic multimeters

In this case we always obtain one of the two results 0 or 1, but errors may appear.[3] We need two-component POVM, A_0, A_1, acting on data and program ($A_0, A_1 \geq 0, A_0 + A_1 = \mathbb{I}$). Optimal multimeter must maximize the mean fidelity (i.e., minimize the error rate) whereas the mean fidelity is defined as follows:

$$F = \int_\psi d\psi \, \frac{1}{2} \mathrm{Tr} \left[|\Psi\rangle\langle\Psi| \, A_0 + |\Psi_\perp\rangle\langle\Psi_\perp| \, A_1 \right] = \mathrm{Tr} \left[R_0 A_0 + R_1 A_1 \right], \quad (1)$$

where the two relevant input states of the multimeter are

$$|\Psi\rangle = |\psi\rangle \otimes |\phi_p(\psi)\rangle, \qquad |\Psi_\perp\rangle = |\psi_\perp\rangle \otimes |\phi_p(\psi)\rangle,$$

with $|\psi\rangle$ and $|\psi_\perp\rangle$ being the basis vectors corresponding to program $|\phi_p(\psi)\rangle$, and

$$R_0 = \frac{1}{2} \int_\psi d\psi \, |\Psi\rangle\langle\Psi|, \qquad R_1 = \frac{1}{2} \int_\psi d\psi \, |\Psi_\perp\rangle\langle\Psi_\perp|. \qquad (2)$$

Integration goes over all orthonormal bases of a qubit.

The optimal POVM that maximizes the mean fidelity must fulfill the following extremal equations:

$$(\lambda - R_0)A_0 = 0, \quad \lambda - R_0 \geq 0, \quad (\lambda - R_1)A_1 = 0, \quad \lambda - R_1 \geq 0, \quad (3)$$

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where λ is a Lagrange multiplier.

This problem is formally equivalent to the determination of the optimal POVM for deterministic (but erroneous) discrimination of two mixed states R_0 and R_1.[4]

2.2 Deterministic multimeter with program $|\psi\rangle^{\otimes N}$

Now let us assume the situation when the program register consists of N copies of the state $|\psi\rangle$. I.e., the program state is $|\phi_p(\psi)\rangle = |\psi\rangle^{\otimes N}$. In this particular situation the optimal POVM has the following form:[3]

$$A_0 = \Pi_+^{(N+1)}, \qquad A_1 = \mathbb{1} - \Pi_+^{(N+1)}, \tag{4}$$

where $\Pi_+^{(N+1)}$ is the projector onto the symmetric subspace of the Hilbert space of $N + 1$ qubits. Notice that this is a projective measurement. This POVM leads to maximal mean fidelity

$$F = \frac{2N + 1}{2N + 2}. \tag{5}$$

In the limit of an infinitely large program register ($N \to \infty$), the ideal projective measurement on the data qubit would be approached ($F \to 1$).

2.3 Optical implementation – an experiment

In this paragraph we will show that the simplest deterministic multimeter with only one program qubit (i.e., $|\phi_p(\psi)\rangle = |\psi\rangle$) can easily be implemented on the ground of experimental quantum optics.[5]

The corresponding POVM on the data and program reads

$$A_0 = \mathbb{1} - A_1, \qquad A_1 = |\Psi^-\rangle\langle\Psi^-|, \tag{6}$$

where

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|1\rangle_p - |1\rangle_d|0\rangle_p).$$

Here the subscripts d and p label the states of data and program qubits, respectively. So, one have to discriminate the singlet from the triplet Bell states. The theoretical value of the mean fidelity is $3/4$.

The qubit states $|0\rangle$ and $|1\rangle$ can be encoded into vertical and horizontal polarizations of photons. One photon represents the *data* and the other one the *program*. It is an easy exercise to show that in the setup displayed in Fig. 1 only the input state $|\Psi^-\rangle$ leads to a coincident detection.

Here are the experimental results for three different program states of the form $\cos\theta\,|V\rangle + \sin\theta\,|H\rangle$:

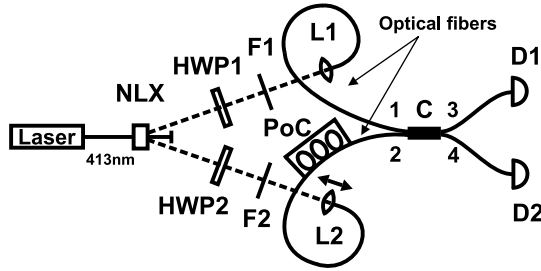| θ	$0°$	$45°$	$90°$
F	0.749	0.694	0.747

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Figure 1. Experimental setup. Notation: NLX – nonlinear crystal, HWP – half-wave plates, F – long-wave pass filters (cut-off at 670 nm), L – lenses, PoC – polarization controller, C – fiber coupler, D – detectors.

The deviation from the theoretical value $3/4$ is mainly due to a non-unit visibility.

2.4 Deterministic multimeter with the program $|\psi\rangle|\psi_\perp\rangle$

Here we will focus on a slightly different program state consisting of two orthogonal states that define the required measurement basis: $|\phi_p(\psi)\rangle = |\psi\rangle|\psi_\perp\rangle$. Now the optimal POVM on data and program reads[3]

$$A_0 = \frac{1}{2}\Pi_+^{(3)} + |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|, \qquad A_1 = \mathbb{1} - A_0, \tag{7}$$

where

$$|\phi_1\rangle = \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)|0\rangle_d|01\rangle_p - (\sqrt{3}-1)|0\rangle_d|10\rangle_p - 2|1\rangle_d|00\rangle_p],$$

$$|\phi_2\rangle = \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)|1\rangle_d|10\rangle_p - (\sqrt{3}-1)|1\rangle_d|01\rangle_p - 2|0\rangle_d|11\rangle_p].$$

The mean fidelity is

$$F' = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) \tag{8}$$

It equals the optimal fidelity of the estimation of $|\psi\rangle$ from a single copy of $|\psi\rangle|\psi_\perp\rangle$. So, the best strategy is to estimate state $|\psi\rangle$ first and then measure the signal qubit in the corresponding basis.

It was shown[6] that the state of two orthogonal qubits $|\psi\rangle|\psi_\perp\rangle$ encodes the information on the state $|\psi\rangle$ better than the state of two identical qubits $|\psi\rangle|\psi\rangle$. One would thus expect that the state $|\psi\rangle|\psi_\perp\rangle$ should also give an advantage when used as a program for the multimeter. Surprisingly, this is not the case and we see that it is *better* to use two *identical* qubits $|\psi\rangle|\psi\rangle$. With such a program we can achieve fidelity $F = 5/6 \approx 0.833$ that is greater than $F' = (1 + 1/\sqrt{3})/2 \approx 0.789$.
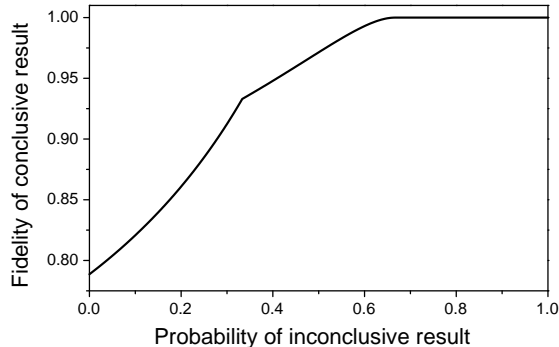
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Figure 2. Dependence of the mean fidelity of conclusive results on the probability of inconclusive result.

2.5 Error-free probabilistic multimeters

Now we will turn our attention to the devices that probabilistically realize exact projective measurement on the data qubit. I.e., they never make an error but from time to time they can give an inconclusive result. We will analyze this case only for program state $|\phi_p(\psi)\rangle = |\psi\rangle|\psi_\perp\rangle$. It can be shown[7] that the optimal three-component POVM on data and program reads

$$A_0 = \frac{2}{3}\left[|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\right], \; A_1 = \frac{2}{3}\left[|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right], \; A_? = \mathbb{1} - A_0 - A_1,$$

$$(9)$$

where

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|01\rangle_p - |1\rangle_d|00\rangle_p), \qquad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|11\rangle_p - |1\rangle_d|10\rangle_p),$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|10\rangle_p - |1\rangle_d|00\rangle_p), \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_d|11\rangle_p - |1\rangle_d|01\rangle_p).$$

The probability of the inconclusive result is $2/3$.

2.6 Multimeters with fixed fraction of inconclusive results

The deterministic multimeters and the error-free probabilistic multimeters can be considered as special limiting cases of a more general class of optimal multimeters that yield inconclusive results but, simultaneously, the fidelity of conclusive results may be lower then one. The optimization task is: For given probability of inconclusive result maximize the mean fidelity.

If program state $|\phi_p(\psi)\rangle = |\psi\rangle|\psi_\perp\rangle$ is assumed then the optimal mean fidelity of conclusive results depends on the probability of inconclusive result in the way depicted in the graph in Fig. 2.[7]

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3 Universal discriminator

Let us suppose that we want to discriminate unambiguously between two known non-orthogonal states of a qubit. However, we would like to have a possibility to "switch" (to program) the apparatus in order to be able to work with several different pairs of states. As a program register we will use another qubit.[1] The two input states of the data qubit that are in correspondence with the program setting may never be wrongly identified (but from time to time we can get an inconclusive result). Let us stress the *quantum nature* of the "programming": The states of the program register that represent different programs can be *non-orthogonal*.

To characterize the pair of states classically one needs an infinite number of bits of classical information. Our procedure requires *only a single qubit* for the same job.

3.1 Universal discriminator – an experiment

Let us consider two input states of a qubit, that should be discriminated, in the following form:

$$|\phi_d^{\pm}\rangle = \alpha\,|V_d\rangle \pm \beta\,|H_d\rangle. \tag{10}$$

The state of a program qubit for the unambiguous discrimination of states (10) is chosen to coincide with $|\phi_d^+\rangle$:

$$|\psi_p\rangle = \alpha\,|V_p\rangle + \beta\,|H_p\rangle. \tag{11}$$

Here $|V\rangle$ and $|H\rangle$ denotes two orthogonal polarization states of photons. So, the total state of the data and program

$$|\phi_d^{\pm}\rangle \otimes |\psi_p\rangle = \sqrt{2}\left[\frac{\alpha^2 \pm \beta^2}{2}\,|\Phi^+\rangle + \frac{\alpha^2 \mp \beta^2}{2}\,|\Phi^-\rangle + \alpha\beta\,|\Psi^{\pm}\rangle\right], \tag{12}$$

where the Bell states are used:

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|V_d\rangle|H_p\rangle \pm |H_d\rangle|V_p\rangle\right), \qquad |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|V_d\rangle|V_p\rangle \pm |H_d\rangle|H_p\rangle\right).$$

Clearly, if we were able to detect Bell states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ we could unambiguously discriminate states $|\phi_d^+\rangle$ and $|\phi_d^-\rangle$. However, we *can* identify these two Bell states by means of coincidence detection – see Fig. 3. If detectors D_1 and D_4 or D_2 and D_3 click together state $|\Psi^-\rangle$ was present in the input (this corresponds to the recognition of state $|\phi_d^-\rangle$). If detectors D_1 and D_2 or D_3 and D_4 click in coincidence the state $|\Psi^+\rangle$ was present in the input (so, $|\phi_d^+\rangle$ is detected). If both photons enter the same detector either $|\Phi^+\rangle$ or $|\Phi^-\rangle$ was present in the input (this represents the inconclusive result of the discrimination). The probability of successful discrimination is $p = 2|\alpha\beta|^2 = 2(|\alpha|^2 - |\alpha|^4)$.
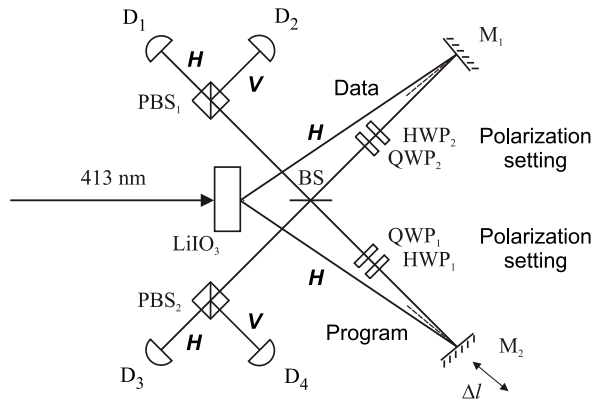
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Figure 3. Universal discriminator. Notation: BS – beam splitter, PBS – polarization beam splitter, HWP – half-wave plate, QWP – quatre-wave plate (they serve for polarization setting), M – mirror, D – detector.

Acknowledgments

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