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Indirect interference of two modes with different frequencies at the one-photon level

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Abstract. The possibility of observing interference with two modes of different frequencies in a one-photon state by means of parametric up-conversion is studied. This phenomenon could be utilized for discernment between pure and mixed states. There is also a close connection to the question of the extent of indefiniteness of the photon's path.

1. Introduction

In recent years, much effort has been paid to the study of various low-intensity optical interference effects on both a theoretical and an experimental basis. Experiments of this kind make it possible to test the foundations of quantum theory and this research plays an important role in the endeavour to understand quantum mechanics on the whole. The experiments performed so far cover, for example single-photon interference experiments (including interference of independent beams [1, 2]) demonstrating, amongst other factors, the wave–particle duality, experiments with fourth-order interference (employing photon pairs generated by frequency down-conversion) which were designed for testing Bell's inequalities, and interference experiments with non-classical light. A comprehensive review with a number of references can be found in [3].

In the present paper we shall deal with the possibility of 'mediating' interference between two modes of different frequencies at the one-photon level. As a motivation for this, we shall show how to use the interference effects to demonstrate the relation between the degree of entanglement and the extent of diagonalization of the density matrix relating to one subsystem. Of course, this is not the only possible application.

Interference experiments can provide information about the off-diagonal elements of the density matrix. For example, let us consider two partially entangled particles, say, two photons with partially correlated polarizations. The fact that the entanglement is imperfect can be found from a coincidence measurement on both particles (two-photon interference), but it also manifests itself in the form of the density matrices describing both subsystems (both photons) separately [4]. A general quantum state of the two photons mentioned above can be written as follows:

$$|\psi\rangle = \alpha|V\rangle_1|V\rangle_2 + \beta|H\rangle_1|H\rangle_2 + \gamma|V\rangle_1|H\rangle_2 + \delta|H\rangle_1|V\rangle_2, \tag{1}$$

where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$ and $|V\rangle, |H\rangle$ represent an orthonormal basis (vertical and horizontal linear polarizations) of each subsystem (the subsystems are differentiated by subscripts 1 and 2). Tracing over the subsystem 2, one obtains the following statistical (or density) operator of the subsystem 1 (in a certain sense, this is a quantum analogue of the classical coherence matrix)

$$\hat{\rho}^{(1)} = \mathrm{Tr}_2 |\psi\rangle\langle\psi| = \sum_{i,j=V,H} \rho_{ij}^{(1)} |i\rangle_1 \langle j|_2, \tag{2}$$

with $\rho_{ij}^{(1)}$ being the corresponding matrix elements:

$$\rho^{(1)} = \begin{bmatrix} |\alpha|^2 + |\gamma|^2 & \alpha\delta^* + \beta^*\gamma \\ \alpha^*\delta + \beta\gamma^* & |\beta|^2 + |\delta|^2 \end{bmatrix} \tag{3}$$

(an asterisk means complex conjugation). A similar result can be obtained for the subsystem 2 by tracing over the subsystem 1.

In general, a state with perfect entanglement can be written as

$$2^{-1/2}\left[|A\rangle_1 |C\rangle_2 + \exp(\mathrm{i}\Omega)\, |B\rangle_1 |D\rangle_2\right],$$

where $|A\rangle_1, |B\rangle_1$ represent an orthonormal basis of the first subsystem (first particle), $|C\rangle_2, |D\rangle_2$ an orthonormal basis of the second subsystem (second particle) and Ω is an arbitrary phase factor. The index of correlation† [4] of such states is maximized $\left(I_C = -2k_B \ln\left(\frac{1}{2}\right)\right)$ and the density matrices describing both subsystems separately differ from the unit matrix only by a number factor:

$$\rho^{(1)} = \rho^{(2)} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \tag{4}$$

This means that they have the same diagonal form in an *arbitrary basis*. If, for example, $|\alpha| \neq |\beta| \neq 0$ and $\gamma = \delta = 0$ in equation (3), then the density matrix is also diagonal but only in our chosen basis‡ [5].

Another extreme case of equation (1) is a general product state

$$(a|V\rangle_1 + b|H\rangle_2)(c|V\rangle_1 + d|H\rangle_2),$$

with $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$. Then

$$\rho^{(1)} = \begin{bmatrix} |ac|^2 + |ad|^2 & ab^* \\ a^*b & |bc|^2 + |bd|^2 \end{bmatrix}. \tag{5}$$

† The degree of entanglement and information carried by density matrices of the subsystems can be well expressed through the *index of correlation* (based on the entropy approach) $I_C = S_1 + S_2 - S$ which, in fact, represents the mutual information. $S_{1(2)} = -k_B \mathrm{Tr}_{1(2)}\left[\hat{\rho}^{1(2)} \ln \hat{\rho}^{1(2)}\right]$ is the entropy of the subsystem 1(2), S is the total entropy and k_B denotes the Boltzmann constant.

‡ In fact, such bases in the subsystems can always be found in which a decomposition of state (1) will contain at most two terms and the density matrices of the subsystems will be diagonal. The existence of these bases follows from Schmidt's theorem.

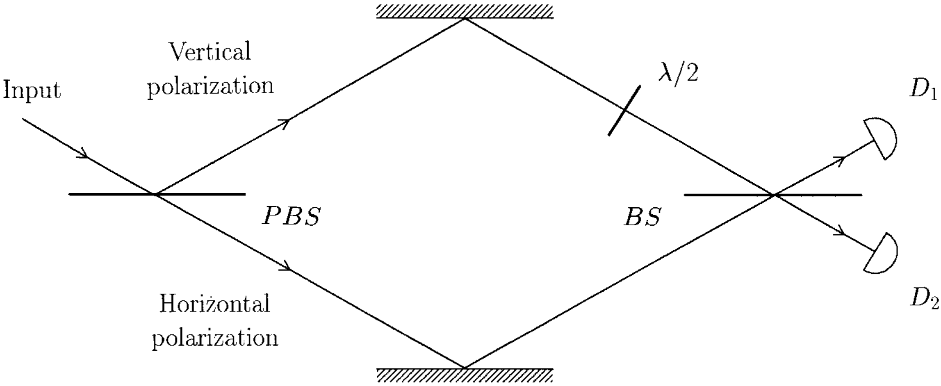

Figure 1. Example of an interference experiment with polarized photons.

If either a or b is zero, then the off-diagonal elements vanish. However, it is only a special case strongly depending on the particular basis.

From the example stated above, it is apparent that the off-diagonal elements of the density matrix describing only one particle of the (possibly entangled) pair can carry some information about the extent of entanglement. There is also a connection with the notion of decoherence caused by a mutual interaction of the subsystems in the past [6]. Thus it may be interesting to measure these off-diagonal elements.

Let us briefly look at the simple interference experiment shown in figure 1. Let us suppose that one photon of the pair (whose polarization state is characterized by the density matrix $\rho^{(1)}$) enters the interferometer. Expressing, in a standard way, the output field operators in terms of the input field operators, including the effect of polarizing beam splitter PBS (in the figure, the vertical polarization is led to the upper arm and the horizontal one to the lower arm), phase shifts in both arms of the interferometer (connected with propagation times), and the effect of the half-wave plate `$\lambda/2$' turning the polarization axis by 90° (which makes possible the interference at the last beam splitter BS), and assuming that light enters only through one input port (in the other the vacuum state is supposed), one can readily find expressions for the photon-number operators $\hat{n}_1$ and $\hat{n}_2$ in the modes entering the detectors D_1 and D_2 and for the corresponding mean numbers of photons. The mean number of photons at D_1 (for example) is

$$n_1 = \tfrac{1}{2} - \left|\rho_{VH}\right| \cos\left[\Delta\,\phi + \arg\left(\rho_{VH}\right)\right], \tag{6}$$

where $\Delta\,\phi$ is the difference between the phase shifts in the lower and upper arms of the interferometer and ρ_{VH} is the density-matrix element (see equation (2)). Thus the visibility reads

$$\mathscr{V}_1 = 2\left|\rho_{VH}\right|. \tag{7}$$

This means that we can measure the modulus of the off-diagonal elements of the density matrix (in principle, it is also possible to find their phases; there is an analogy with the reconstruction of the correlation function in the Young two-slit interference experiment [7]). The measurements can be performed in various bases, for example rotating the apparatus from figure 1 appropriately, we can change between different linear-polarization bases. If the particle pertains to a perfectly entangled pair, there is no interference in any basis.

It is easy to provide conditions for the interference of two modes with different linear polarizations. We simply turn the polarization axis of one of them by means of a $\lambda/2$ plate. There is no measurable energy exchange in this process and we cannot learn which path the photon has taken. Another situation occurs when we are dealing with modes differing in frequencies. For example, photon pairs produced by frequency down-conversion exhibit just energy (frequency) correlations [8, 9]. Here, modes of many different frequencies are always present. However, in the case of perfect energy correlation (defined (for a pair of photons) by

$$\sum_{\omega} \eta(\omega)|\omega\rangle_1|\omega_0 - \omega\rangle_2, \tag{8}$$

where $\eta(\omega)$ is some properly normalized function, ω_0 is a constant (pumping frequency), and $|\omega\rangle_1, |\omega\rangle_2$ represent the eigenstates of the free Hamiltonians of the subsystems corresponding to energy $\hbar\omega$) the density matrix of each subsystem expressed in energy representation is always diagonal. This form corresponds to a statistical ensemble of individual monochromatic photons with probability distribution $|\eta(\omega)|^2$ (or $|\eta(\omega_0 - \omega)|^2$ for the other subsystem). On the contrary, when the pair is in a product state, there are always some non-zero off-diagonal elements in the density matrices of the subsystems (in this case each subsystem can be described by a pure state). Thus the interference measurement of the off-diagonal elements could again give some notion of the extent of entanglement. How do we change the mode of frequency ω_1 to another mode of some different frequency ω_2 without allowing `which-path information' to be read and thereby destroying interference? In the next section we shall try to find at least a partial answer to such a question.

2. Interferometer with frequency conversion

We shall study an interferometer with a frequency conversion device in one arm as shown in figure 2. For simplicity we shall assume the input state (entering

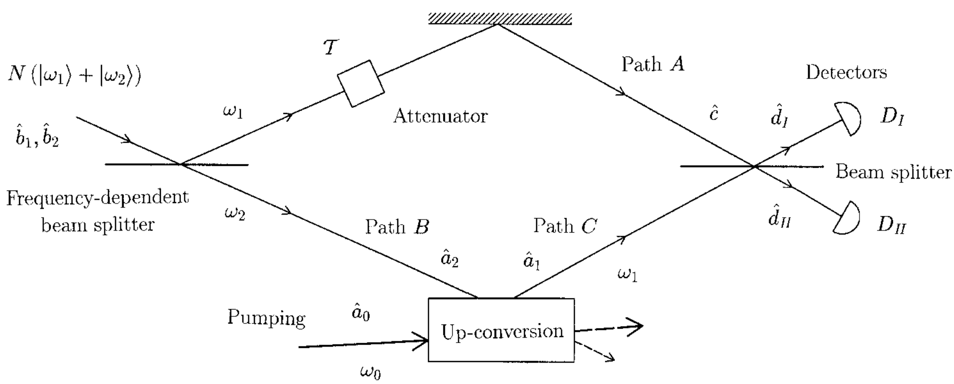

Figure 2. Outline of the proposed experimental arrangement.

one port of the first beam splitter) as a pure *one-photon* state containing only two modes with frequencies ω_1 and ω_2:

$$N(|\omega_1\rangle + |\omega_2\rangle); \tag{9}$$

N is a normalization constant. It is also assumed that $\omega_1 > \omega_2$. At the other input the vacuum state is supposed. The input field is divided at the first frequency-dependent beam splitter so that the component of frequency ω_1 is reflected to the upper arm and the component with frequency ω_2 is transmitted to the lower arm of the interferometer. In the lower arm, there is a nonlinear crystal pumped at frequency ω_0. Within the crystal, the frequency up-conversion process occurs. Let the frequency of the pump beam be such that the up-converted radiation has just the frequency ω_1, that is

$$\omega_0 = \omega_1 - \omega_2.$$

Further, let us assume perfect phase matching. (Of course, it affects the geometry of the experimental arrangement but, to keep lucidity, this is not fully respected in the scheme in figure 2.) Under these conditions, the interaction Hamiltonian for the three-mode parametric interaction in the interaction picture takes the form [10]

$$H_{\mathrm{I}} = \hbar g \hat{a}_1^{\dagger} \hat{a}_2 \hat{a}_0 + \mathrm{Hc}, \tag{10}$$

where g is a real coupling constant, $\hat{a}_j$ and $\hat{a}_j^{\dagger}$ $(j = 0, 1, 2)$ are the annihilation and creation operators of the corresponding modes respectively, and Hc denotes Hermitian conjugation. Then the unitary time evolution operator is

$$U(t, t - \tau) = \exp\left(-\frac{\mathrm{i}}{\hbar} H_{\mathrm{I}} \tau\right), \tag{11}$$

with τ being an interaction time. What does the interaction time mean here? The crystal may be mounted in the experimental set-up for rather a long time. However, in practice we hardly meet ideal monochromatic states or their simple combinations. Usually we dispose of localized wave packets and such wave packets spend only some delimited time inside the crystal. This finite time interval can be regarded as an effective duration of the interaction. Thus, even though we shall work with the idealized model with three monochromatic modes, we shall assume a finite effective interaction time τ. Now we expand the exponential in equation (11) and retain only the first two terms of the expansion and neglect all the others, supposing that the conditions for doing so (short effective interaction time and sufficiently low efficiency of the process) are fulfilled. This means that we shall neglect low probable backward processes and any other more complicated transitions. The approximated evolution operator then reads

$$U(t, t - \tau) \approx \hat{1} - \mathrm{i}(g\tau \hat{a}_1^{\dagger} \hat{a}_2 \hat{a}_0 + \mathrm{Hc}). \tag{12}$$

We start with the input state given by equation (9). Let us further assume that the mode 1 leaving the up-converter via the path C is not populated at the beginning, that is there is a field vacuum there. We shall study the interference at the output beam splitter of the interferometer in two cases: firstly when the up-converter is pumped by a field in a number state, and secondly when it is pumped by a field in a coherent state. It seems that, in case of the number-state pumping, it is possible to find out (monitoring the number of photons in the pump beam) whether the up-conversion process has occurred or not, that is to get which-path information. Therefore no interference should be expected. However, uncertainty in the photon number in the coherent state raises hopes to observe some interference.

Let us designate the annihilation operators of the input field corresponding to two modes with frequencies ω_1 and ω_2 (incoming from the upper side of the first beam splitter; see figure 2) as $\hat{b}_1$ and $\hat{b}_2$ respectively. The annihilation operators related to the other (unused) input port will be designated as $\hat{e}_1$ and $\hat{e}_2$ (for ω_1 and ω_2 respectively). Further, let $\hat{c}$ denote the annihilation operator of the field (of frequency ω_1) entering the second beam splitter from the path A and let $\hat{d}_{\mathrm{I}}$ and $\hat{d}_{\mathrm{II}}$ be the annihilation operators of the modes exiting the interferometer and immediately entering the corresponding detectors. Of course, $\hat{a}_0$, $\hat{a}_1$ and $\hat{a}_2$ correspond to modes entering or exiting the up-conversion crystal and having frequencies ω_0, ω_1 and ω_2 respectively. The attenuator can be regarded as a beam splitter with amplitude reflectance $\mathscr{R}$ and transmittance $\mathscr{T}$ (which will be assumed to be real). The annihilation operator corresponding to the mode at its unused input port will be labelled $\hat{f}$ (in this port the vacuum state is supposed).

Thus, if the first beam splitter ideally divides both frequency components, we can express the annihilation operators $\hat{c}$ and $\hat{a}_2$ in terms of $\hat{b}_1, \hat{b}_2$ and $\hat{e}_1, \hat{e}_2, \hat{f}$:

$$\hat{c} = \mathrm{i}\mathscr{R} \exp\left(\mathrm{i}\phi_{\hat{A}}\right)\hat{f} + \mathscr{T} \exp\left(\mathrm{i}\phi_{\mathrm{A}}\right)\left(\mathrm{i}\hat{b}_1 + \hat{e}_1\right), \tag{13}$$

$$\hat{a}_2 = \exp\left(\mathrm{i}\phi_{\mathrm{B}}\right)\left(\hat{b}_2 + \mathrm{i}\hat{e}_2\right). \tag{14}$$

Here $\phi_{\mathrm{A}} = (\omega_1/c)\ell_{\mathrm{A}}$ and $\phi_{\mathrm{B}} = (\omega_2/c)\ell_{\mathrm{B}}$ are the phase shifts associated with the propagation along the path A (of the length ℓ_{A}) and B (of the length ℓ_{B}) respectively (see figure 2); $\phi_{\hat{A}} = (\omega_1/c)\ell_{\hat{A}}$, where $\ell_{\hat{A}}$ is the distance between the attenuator and the second beam splitter. Similarly, assuming the second beam splitter to be lossless with splitting ratio 50%: 50%, one can express the detector-mode operators $\hat{d}_{\mathrm{I}}$ and $\hat{d}_{\mathrm{II}}$ in the following way:

$$\hat{d}_{\mathrm{I}} = \frac{1}{2^{1/2}}\left[\mathrm{i}\hat{c} + \exp\left(\mathrm{i}\phi_{\mathrm{C}}\right)\hat{a}_1\right], \tag{15}$$

$$\hat{d}_{\mathrm{II}} = \frac{1}{2^{1/2}}\left[\hat{c} + \mathrm{i}\exp\left(\mathrm{i}\phi_{\mathrm{C}}\right)\hat{a}_1\right], \tag{16}$$

where $\phi_{\mathrm{C}} = (\omega_2/c)\ell_{\mathrm{C}}$ is the phase shift associated with the path C. Using the annihilation operators $\hat{d}_{\mathrm{I}}$ and $\hat{d}_{\mathrm{II}}$ (and the corresponding creation operators), one can define the mean numbers of photons n_{I} and n_{II} detected in detectors $\mathrm{D_I}$ and $\mathrm{D_{II}}$ respectively. (If only one frequency component is present, these quantities may substitute the more general detection rates. In our case, when only one photon is in the system, these quantities are also proportional to the probabilities of detecting the photon at the first or at the second detector.) The related photon-number operators are

$$\hat{n}_{\mathrm{I}} = \hat{d}_{\mathrm{I}}^{\dagger}\hat{d}_{\mathrm{I}} = \tfrac{1}{2}\Big\{\hat{b}_1^{\dagger}\hat{b}_1\mathscr{T}^{\,2} + \hat{a}_1^{\dagger}\hat{a}_1 - \hat{b}_1^{\dagger}\hat{a}_1\mathscr{T}\exp\left[-\mathrm{i}(\phi_{\mathrm{A}} - \phi_{\mathrm{C}})\right] - \hat{a}_1^{\dagger}\hat{b}_1\mathscr{T}\exp\left[\mathrm{i}(\phi_{\mathrm{A}} - \phi_{\mathrm{C}})\right]$$
$$+ \text{ terms containing } \hat{e}_1^{\dagger} \text{ or } \hat{f}^{\dagger} \text{ on the left and/or } \hat{e}_1 \text{ or } \hat{f} \text{ on the right}\Big\} \tag{17}$$

and

$$\hat{n}_{\mathrm{II}} = \hat{d}_{\mathrm{II}}^{\dagger}\hat{d}_{\mathrm{II}} = \tfrac{1}{2}\Big\{\hat{b}_1^{\dagger}\hat{b}_1\mathscr{T}^{\,2} + \hat{a}_1^{\dagger}\hat{a}_1 + \hat{b}_1^{\dagger}\hat{a}_1\mathscr{T}\exp\left[-\mathrm{i}(\phi_{\mathrm{A}} - \phi_{\mathrm{C}})\right] + \hat{a}_1^{\dagger}\hat{b}_1\mathscr{T}\exp\left[\mathrm{i}(\phi_{\mathrm{A}} - \phi_{\mathrm{C}})\right]$$
$$+ \text{ terms containing } \hat{e}_1^{\dagger} \text{ or } \hat{f}^{\dagger} \text{ on the left and/or } \hat{e}_1 \text{ or } \hat{f} \text{ on the right}\Big\} \tag{18}$$

The terms containing the operators of modes e_1 and f are not stated here explicitly because, assuming the vacuum state at the appropriate inputs, they clearly give no contribution to the mean photon numbers calculated below.

The visibility is defined by the formula

$$\mathscr{V}_{\mathrm{i}} = \frac{n_{\mathrm{i\,max}} - n_{\mathrm{i\,min}}}{n_{\mathrm{i\,max}} + n_{\mathrm{i\,min}}}, \tag{19}$$

where $\mathrm{i} = \mathrm{I}, \mathrm{II}$ and the minimal and maximal values are taken with respect to the phase-shift variables.

First, we shall assume the pumping of the nonlinear medium in the form of a number state (Fock state). The field entering the interferometer is considered in the form of a superposition of two one-photon states corresponding to two different frequencies, as we have stated above (for simplicity, only discrete values of frequency are considered now so that the normalization constant N from equation (9) is equal to $2^{-1/2}$). Thus the state vector describing the initial state of the field (at time $t = 0$) can be expressed as

$$|\psi_{\mathrm{f}}(0)\rangle = \frac{1}{2^{1/2}}(\hat{b}_1^{\dagger} + \hat{b}_2^{\dagger})\frac{(\hat{a}_0^{\dagger})^n}{(n!)^{1/2}}|\mathrm{vac}\rangle. \tag{20}$$

Owing to the interaction it develops into the state vector (we are still working in the interaction picture)

$$|\psi_{\mathrm{f}}(\tau)\rangle = U(\tau, 0)|\psi_{\mathrm{f}}(0)\rangle \approx |\psi_{\mathrm{f}}(0)\rangle - \mathrm{i}g\tau\exp(\mathrm{i}\phi_{\mathrm{B}})\frac{1}{2^{1/2}}\frac{n^{1/2}}{[(n-1)!]^{1/2}}\hat{a}_1^{\dagger}(\hat{a}_0^{\dagger})^{n-1}|\mathrm{vac}\rangle. \tag{21}$$

Here we made use of the commutation relations of creation and annihilation operators and of the fact that any annihilation operator acting on the vacuum state gives zero. It is evident from equation (21) that, measuring the number of photons in the pump beam behind the crystal, one may obtain either n or $n - 1$. The latter result indicates that the up-conversion has occurred. This enables us, at least in principle, to discern by which path the photon came to the last beam splitter of the interferometer. The mean number of photons impinging on the detector D_{I} is

$$n_{I}^{(f)} = \langle \psi_f(\tau)|\hat{n}_I|\psi_f(\tau)\rangle = \tfrac{1}{4}(\mathscr{T}^2 + ng^2\tau^2). \qquad (22)$$

This expression is independent of any phase shifts at any arm of the interferometer. Thus the visibility is zero. For the detector D_{II}, the results are the same. No interference appears as we have expected.

Now we shall analyse the situation when the pump field is in a coherent state of a complex amplitude α. Let the field entering the interferometer be the same as in the previous case. The corresponding initial state vector is given by the formula

$$|\psi_c(0)\rangle = \frac{1}{2^{1/2}}(\hat{b}_1^\dagger + \hat{b}_2^\dagger)\exp\left(\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\hat{a}_0^\dagger)^n|\text{vac}\rangle. \qquad (23)$$

The final state after the interaction is then

$$\begin{aligned}|\psi_c(\tau)\rangle &= U(\tau,0)|\psi_c(0)\rangle \\ &\approx |\psi_c(0)\rangle - \mathrm{i}g\tau\alpha\exp(\mathrm{i}\phi_B)\exp\left(\frac{|\alpha|^2}{2}\right)\frac{1}{2^{1/2}}\hat{a}_1^\dagger\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}(\hat{a}_0^\dagger)^n|\text{vac}\rangle. \qquad (24)\end{aligned}$$

This form was obtained employing the fact that the coherent state is an eigenstate of the annihilation operator. Using equations (17), (18) and (24), one can express the mean numbers of photons registered by (ideal) detectors D_{I} and D_{II}:

$$n_{I}^{(c)} = \langle \psi_c(\tau)|\hat{n}_I|\psi_c(\tau)\rangle = \tfrac{1}{4}\left[\mathscr{T}^2 + |\alpha|^2g^2\tau^2 - 2|\alpha|\mathscr{T}g\tau\cos(\Delta\phi - \theta)\right], \qquad (25)$$

$$n_{II}^{(c)} = \langle \psi_c(\tau)|\hat{n}_{II}|\psi_c(\tau)\rangle = \tfrac{1}{4}\left[\mathscr{T}^2 + |\alpha|^2g^2\tau^2 + 2|\alpha|\mathscr{T}g\tau\cos(\Delta\phi - \theta)\right], \qquad (26)$$

where $\Delta\phi = \phi_A - \phi_B - \phi_C$ and $\theta = \arg(\alpha)$. Clearly, the interference appears now. Substituting equations (25) and (26) into equation (19), we obtain the visibilities†

$$\mathscr{V}_{I}^{(c)} = \mathscr{V}_{II}^{(c)} = \frac{2|\alpha|\mathscr{T}g\tau}{\mathscr{T}^2 + |\alpha|^2g^2\tau^2}. \qquad (27)$$

One can even notice that, if $\mathscr{T} = |\alpha|g\tau$, then $\mathscr{V}_{I}^{(c)} = \mathscr{V}_{II}^{(c)} = 1$. That is, introducing proper losses into the upper arm of the interferometer, the visibility may be increased up to unity (of course, at the expense of decreasing the mean number of photons). In the case of coherent pumping, the behaviour of the quantum system is similar to the case of classical waves.

The higher terms in the expansion of the evolution operator given by equation (11) do not influence the crucial fact of the occurrence or absence of interference in the cases presented above.

The wave–particle duality is manifested here. Any uncertainty in the number of photons in the pump beam always induces uncertainty in the which-path

† If the examined input state contained, in addition, the vacuum state in coherent superposition, the visibility would not be changed; only the mean numbers of photons would be reduced by some factor.

information concerning the photon in the interferometer, and interference appears. It is discussed in more detail in section 3.

From the practical point of view, the coherent-state pumping is of interest; it is experimentally feasible and it may be utilized for the purposes mentioned in the previous section, that is for the measurement of the off-diagonal elements of the density matrix of the input field, as will be shown in section 4.

3. The effect of pump-field statistics

In order to demonstrate how the uncertainty of the photon number of the pump field invokes interference, let us consider a very simple example. Let the pump field be in a superposition of the one-photon and zero-photon state (in the mode 0 of frequency ω_0) and let the input field be in the same state as introduced above. Thus

$$|\psi(0)\rangle = \frac{1}{2^{1/2}}(\hat{b}_1^\dagger + \hat{b}_2^\dagger)(\alpha\hat{1} + \beta\hat{a}_0^\dagger)|\text{vac}\rangle, \tag{28}$$

with $|\alpha|^2$ and $|\beta|^2$ represent the probabilities of finding no photon or one photon respectively, and, of course, $|\alpha|^2 + |\beta|^2 = 1$. After interaction, the system comes to the state

$$|\psi(\tau)\rangle \approx |\psi(0)\rangle - \mathrm{i}g\tau\beta\exp(\mathrm{i}\phi_\mathrm{B})\frac{1}{2^{1/2}}\hat{a}_1^\dagger|\text{vac}\rangle. \tag{29}$$

Then the mean number of photons detected in detector $\mathrm{D_I}$ is

$$\langle\psi(\tau)|\hat{n}_\mathrm{I}|\psi(\tau)\rangle = \tfrac{1}{4}\left[\mathcal{T}^2 + |\beta|^2 g^2\tau^2 - 2|\alpha\beta|\mathcal{T}g\tau\cos(\Delta\phi + \theta)\right], \tag{30}$$

where $\theta = \arg(\alpha\beta^*)$. (The result for the detector $\mathrm{D_{II}}$ differs only in the sign of the last term.) Thus the visibility (for both detectors) is

$$\mathcal{V} = \frac{2|\alpha\beta|\mathcal{T}g\tau}{\mathcal{T}^2 + |\beta|^2 g^2\tau^2} = \frac{2(|\beta|^2 - |\beta|^4)^{1/2}\mathcal{T}g\tau}{\mathcal{T}^2 + |\beta|^2 g^2\tau^2} = \frac{2\sigma_n\mathcal{T}g\tau}{\mathcal{T}^2 + \bar{n}g^2\tau^2}, \tag{31}$$

where $\sigma_n^2 = \langle(\hat{n}_0 - \langle\hat{n}_0\rangle)^2\rangle$ is the photon number variance of the pump field and $\bar{n} = \langle\hat{n}_0\rangle$ is its mean photon number. For example, if the attenuator is set so that $\mathcal{T} = g\tau$, then the visibility takes a simple form

$$\mathcal{V} = \frac{2(|\beta|^2 - |\beta|^4)^{1/2}}{1 + |\beta|^2} = \frac{2\sigma_n}{1 + \bar{n}}, \tag{32}$$

which is plotted in figure 3.

However, in general, the visibility cannot be expressed as a function of the photon-number variance of the pump beam. Let us suppose the pumping to be in some general (arbitrary) quantum state, so that

$$|\psi(0)\rangle = \frac{1}{2^{1/2}}(|b_1\rangle + |b_2\rangle)|\text{pump}\rangle|\text{vac}_\text{rest}\rangle, \tag{33}$$

where $|b_i\rangle = \hat{b}_i^\dagger|\text{vac}_{1,2}\rangle$, $i = 1, 2$, represent one-photon states of the input field

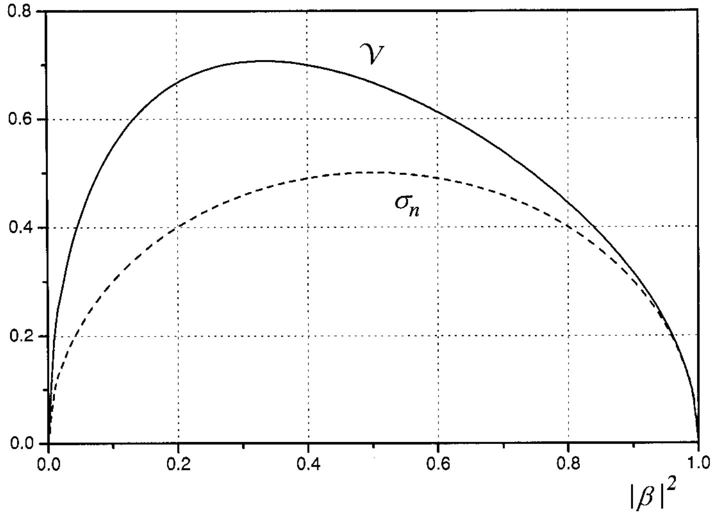

Figure 3. Relation between the visibility and the uncertainty of the photon number of the pump beam: simple example.

corresponding to the modes of frequencies ω_1 and ω_2, and $|\text{pump}\rangle$ denotes the state of the pump field (the mode 0). Other modes (e_1, e_2, f and a_1) are assumed to be in the vacuum state $(|\text{vac}_{\text{rest}}\rangle)$ at the beginning. Calculating, in the usual way, the state $|\psi(\tau)\rangle$, one can express the mean number of photons at the detector D_I (or, by analogy, at D_{II})

$$\langle\psi(\tau)|\hat{n}_I|\psi(\tau)\rangle = \tfrac{1}{4}\{\mathcal{T}^2 + \bar{n}g^2\tau^2 - 2|\xi|\mathcal{T}g\tau\cos[\Delta\phi + \arg(\xi)]\}. \qquad (34)$$

Here $\bar{n} = \langle\text{pump}|\hat{a}_0^\dagger\hat{a}_0|\text{pump}\rangle$ and $\xi = \langle\text{pump}|\hat{a}_0^\dagger|\text{pump}\rangle$. Then the visibility is given by the formula

$$\mathcal{V} = \frac{2|\xi|\mathcal{T}g\tau}{\mathcal{T}^2 + \bar{n}g^2\tau^2}. \qquad (35)$$

4. Measurement of off-diagonal elements of the density matrix

Now we shall show how to employ the knowledge obtained in section 2 to determine the off-diagonal elements of the density matrix of the input field occurring in a mixed state consisting of two modes of different frequencies (by analogy to the previously mentioned case with the polarization states).

The input field will be assumed in some general (*mixed*) one-photon state consisting of two modes with frequencies ω_1 and ω_2 (as the basis in this subspace we use the states $|b_i\rangle = \hat{b}_i^\dagger|\text{vac}_{1,2}\rangle$; $i = 1, 2$). The pump field (the mode 0 of frequency ω_0) will be assumed to be in a coherent state $|\alpha\rangle$ of the complex amplitude α. No other modes will be populated at the beginning $(|\text{vac}_{\text{rest}}\rangle)$. This state of the system can be described by the following statistical operator:

$$\hat{\rho}(0) = \sum_{i,j=1,2} \rho_{ij}|b_i\rangle\langle b_j| \otimes |\alpha\rangle\langle\alpha| \otimes |\text{vac}_{\text{rest}}\rangle\langle\text{vac}_{\text{rest}}|, \qquad (36)$$

where ρ_{ij} are matrix elements characterizing the input field. From the facts that the density operator has a unit trace and that it is Hermitian, it follows that $\rho_{11} + \rho_{22} = 1$ and $\rho_{ij} = \rho_{ji}^*$. The first expression can be understood as that the total probability of finding a photon in either of the two modes is unity. Because of time evolution, the density operator after the interaction takes the form

$$\hat{\rho}(\tau) = \hat{U}(\tau, 0)\hat{\rho}(0)\hat{U}^\dagger(\tau, 0). \tag{37}$$

The mean number of photons impinging on the detector D_I is then given by the formula

$$\begin{aligned} n_I &= \mathrm{Tr}\left[\hat{\rho}(\tau)\hat{n}_I\right] \\ &\approx \tfrac{1}{2}\left\{\rho_{11}\mathscr{T}^2 + \rho_{22}|\alpha|^2 g^2\tau^2 - 2|\rho_{12}||\alpha|\mathscr{T} g\tau \cos\left[\Delta\phi - \arg(\alpha) + \arg(\rho_{12})\right]\right\}. \end{aligned} \tag{38}$$

The visibility (both at the detector D_I and D_{II}) then reads

$$\mathscr{V} = \frac{2|\rho_{12}||\alpha|\mathscr{T} g\tau}{\rho_{11}\mathscr{T}^2 + \rho_{22}|\alpha|^2 g^2\tau^2}. \tag{39}$$

It is evident that adjusting the attenuator properly so that $\mathscr{T} = |\alpha| g\tau$, the expression for the visibility essentially simplifies to

$$\mathscr{V} = 2|\rho_{12}|. \tag{40}$$

This expression is rather similar to equation (7) and it enables us to determine the modulus of the off-diagonal elements of the density matrix describing the input field.

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