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Programmable quantum measurement device that approximates all projective measurements on a qubit

Jaromír Fiurášek, Miloslav Dušek*, and **Radim Filip**

Department of Optics, Palacký University, 17. listopadu 50, 772 00 Olomouc, Czech Republic

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A "universal" quantum measurement device is proposed. It can approximate any projective measurement on a qubit. The measurement basis is selected by the quantum state of a "program register". The device is optimized with respect to maximal average fidelity. A contra-intuitive result is that if two qubits in the same state are used as a program the average fidelity is higher than if the second program qubit is in the respective orthogonal state.

1 Introduction

Recently, the notion of quantum "multimeters" was introduced [1]. Such programmable quantum devices can realize (either exactly or at least approximately) any desired generalized quantum measurement from a chosen set. Their main feature is that the particular positive operator valued measure (POVM) is selected by the quantum state of a "program register" (quantum software). In this sense they are analogous to universal quantum processors [2–4]. Quantum multimeters could play an important role in quantum state estimation and quantum information processing.

In this contribution, we will describe a programmable quantum device that can approximately accomplish *any* projective von Neumann measurement on a single qubit. Since it is impossible to encode an arbitrary unitary operation (acting on a finite-dimensional Hilbert space) into a state of a finite-dimensional quantum system [2] it is also impossible to encode arbitrary projective measurement on a qubit into such a state [1]. However, it is still possible to encode POVM's that represent, in a certain sense, the best approximation of the required projective measurements.

2 Theoretical description

Suppose we would like to measure a qubit in the basis represented by two orthogonal vectors $|\psi\rangle$ and $|\psi_\perp\rangle$. We want this measurement basis be controlled by the quantum state of a program register, $|\phi_P(\psi)\rangle$. An *ideal* multimeter should give two results, 0 and 1, according to the following prescription:

$$|\psi\rangle \otimes |\phi_{\mathrm{p}}(\psi)\rangle \to 0, \qquad |\psi_\perp\rangle \otimes |\phi_{\mathrm{p}}(\psi)\rangle \to 1. \tag{1}$$

As mentioned above, such a measurement cannot be implemented exactly for all ψ's. Thus, our task is to find such a POVM (acting on the "signal" qubit and the program register together) that would represent the *closest approximation* to our demand. We focus on the scenario when we always obtain one of the two

* Corresponding author E-mail: dusek@optnw.upol.cz

results 0 or 1, but errors may appear. Our aim is to minimize the probability of error, i.e., we will maximize the probability of the correct discrimination between states $|\psi\rangle$ and $|\psi_\perp\rangle$. In general we could optimize both the program and the fixed POVM so as to optimally approximate map (1) for a given dimension of the program register. However, this is an extremely hard problem that we will not attempt to solve in its generality. Instead, we will optimize the fixed POVM for two natural choices of the program.

First we will assume that the program register contains N copies of the state $|\psi\rangle$, $|\phi_{\rm p}(\psi)\rangle = |\psi\rangle^{\otimes N}$. Our second choice of the program – the two-qubit state $|\psi\rangle|\psi_\perp\rangle$ – is motivated by recent results on optimum quantum state estimation. Gisin and Popescu showed that the state of two orthogonal qubits $|\psi\rangle|\psi_\perp\rangle$ encodes the information on the state $|\psi\rangle$ better than state of two identical qubits $|\psi\rangle|\psi\rangle$ [5]. If we possess one copy of the state $|\psi\rangle|\psi_\perp\rangle$ than we can estimate $|\psi\rangle$ with fidelity $\mathcal{F}_\perp = (1+1/\sqrt{3})/2 \approx 0.7886$ which is slightly higher than the fidelity of optimum estimation on one copy of two identical qubits, $\mathcal{F}_{||} = 3/4$ [5, 6]. One would thus expect that the state $|\psi\rangle|\psi_\perp\rangle$ should also give an advantage when used as a program of the multimeter. Rather surprisingly, this is not the case and we shall see that it is better to use two identical qubits $|\psi\rangle|\psi\rangle$.

3 Optimal POVM, the first program choice

Let us define the fidelity $F(\psi)$ of our multimeter (projecting onto states $|\psi\rangle$ and $|\psi_\perp\rangle$) as the probability that a correct measurement result will be obtained when we send the states $|\psi\rangle$ or $|\psi_\perp\rangle$ to the input randomly each with probability one half. This probability can be interpreted as a success rate of the discrimination between two orthogonal states $|\psi\rangle$ and $|\psi_\perp\rangle$.

$$F(\psi) = \frac{1}{2}\mathrm{Tr}\left[|\Psi\rangle\langle\Psi|\, A_0 + |\Psi_\perp\rangle\langle\Psi_\perp|\, A_1\right]. \tag{2}$$

Assuming the program state to be $|\psi\rangle^{\otimes N}$, the two relevant input states of the multimeter read

$$|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle^{\otimes N}, \qquad |\Psi_\perp\rangle = |\psi_\perp\rangle \otimes |\psi\rangle^{\otimes N}. \tag{3}$$

The two components of the searched POVM that should be optimized are denoted as A_0, A_1,

$$A_0, A_1 \geq 0, \qquad A_0 + A_1 = \mathbf{1}. \tag{4}$$

The figure of merit that we would like to maximize is the mean fidelity obtained on averaging $F(\psi)$ over all pure qubit states $|\psi\rangle$, i.e., over the surface of the Bloch sphere,

$$F = \int_\psi d\psi\, F(\psi) = \mathrm{Tr}\left[R_+ A_0 + R_- A_1\right], \tag{5}$$

where the operators R_+ and R_- are given by the integrals

$$R_+ = \frac{1}{2}\int_\psi d\psi\, |\Psi\rangle\langle\Psi|, \qquad R_- = \frac{1}{2}\int_\psi d\psi\, |\Psi_\perp\rangle\langle\Psi_\perp|. \tag{6}$$

A straightforward calculation reveals that R_+ is proportional to the projector onto symmetric subspace $\mathcal{H}_+^{(N+1)}$ of the Hilbert space of $N+1$ qubits,

$$R_+ = \frac{1}{2(N+2)}\Pi_+^{(N+1)}. \tag{7}$$

Furthermore, the sum of operators R_+ and R_- is proportional to the identity operator, $R_+ + R_- = \mathbf{1}/[2(N+1)]$. Thus we immediately have

$$R_- = \frac{1}{2(N+1)}\mathbf{1} - \frac{1}{2(N+2)}\Pi_+^{(N+1)}. \tag{8}$$

The determination of the optimum POVM amounts to maximization of the linear function (5) under the constraints (4). The optimum joint generalized quantum measurement on the signal qubit and N program qubits that maximizes the mean fidelity (5) must satisfy the following extremal equations [7]

$$\begin{aligned} (\lambda - R_+)A_0 &= 0, \qquad & \lambda - R_+ &\geq 0, \\ (\lambda - R_-)A_1 &= 0, \qquad & \lambda - R_- &\geq 0, \end{aligned} \tag{9}$$

where λ is a positive-definite operator. The extremal equations can be very efficiently solved numerically by means of repeated iterations (see, e.g., [8]). From the numerical results, we were able to conjecture the optimum POVM,

$$A_0 = \Pi_+^{(N+1)}, \qquad A_1 = \Pi_-^{(N+1)}, \tag{10}$$

where $\Pi_-^{(N+1)} = \mathbf{1} - \Pi_+^{(N+1)}$ is a projector onto subspace orthogonal to the symmetric subspace of $N+1$ qubits. Taking into account condition (4) it is easy to check that extremal equations (9) are really satisfied. In particular one can find after some algebra that

$$\lambda = \frac{1}{2(N+1)}\mathbf{1} - \frac{1}{2(N+1)(N+2)}\Pi_+^{(N+1)}. \tag{11}$$

On inserting the expressions (7), (8), and (10) into Eq. (5), we obtain the mean fidelity

$$F = \frac{2N+1}{2N+2}. \tag{12}$$

We can now determine the effective POVM carried out on the signal qubit,

$$\begin{aligned} \Pi_{||} &= \mathrm{Tr_p}\left[\mathbf{1}_\mathrm{s} \otimes (|\psi\rangle\langle\psi|)^{\otimes N}\, \Pi_+^{(N+1)}\right], \\ \Pi_{\perp} &= \mathrm{Tr_p}\left[\mathbf{1}_\mathrm{s} \otimes (|\psi\rangle\langle\psi|)^{\otimes N}\, \Pi_-^{(N+1)}\right], \end{aligned} \tag{13}$$

where $\mathrm{Tr_p}$ denotes trace over the program qubits. The outcome $\Pi_\perp$ cannot occur if the input state is $|\psi\rangle$ because the input state $|\Psi\rangle$ belongs to the symmetric subspace of $N+1$ qubits and $\Pi_{||}$ clicks with certainty. Hence the POVM element $\Pi_\perp$ must be proportional to the projector $|\psi_\perp\rangle\langle\psi_\perp|$. Since the sum of POVM elements (13) is an identity operator, we have the following ansatz,

$$\begin{aligned} \Pi_{||} &= |\psi\rangle\langle\psi| + (1-p)|\psi_\perp\rangle\langle\psi_\perp|, \\ \Pi_\perp &= p\,|\psi_\perp\rangle\langle\psi_\perp|. \end{aligned} \tag{14}$$

The probability p that $\Pi_\perp$ clicks when the input state is $|\psi_\perp\rangle$ is given by $p = \langle\Psi_\perp|\Pi_-^{(N+1)}|\Psi_\perp\rangle$. After some algebra we get $p = N/(N+1)$ and the effective POVM representing our universal multimeter reads

$$\begin{aligned} \Pi_{||} &= \frac{1}{N+1}\mathbf{1} + \frac{N}{N+1}|\psi\rangle\langle\psi|, \\ \Pi_\perp &= \frac{N}{N+1}|\psi_\perp\rangle\langle\psi_\perp|. \end{aligned} \tag{15}$$

Notice that the POVM (15) is asymmetric, which reflects the asymmetry of the program register. Furthermore, the fidelity $F(\psi)$ is independent of ψ and equal to the mean fidelity (12). In the limit of infinitely large program register ($N \to \infty$), POVM (15) approaches the ideal projective measurement ($F \to 1$).

4 Optimal POVM, the second program choice

Now let us turn our attention to the program $|\psi\rangle|\psi_\perp\rangle$. For this program state the optimum joint POVM on the signal qubit and the program register can be found following the same procedure as described above for the program $|\psi\rangle^{\otimes N}$. Briefly, one has to calculate operators R_+ and R_- and solve extremal equations (9). We will not give the details of calculations here and only present the results. The two elements of this three-qubit POVM read

$$\begin{aligned} \Pi_+ &= \frac{1}{2}\Pi_+^{(3)} + |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|, \\ \Pi_- &= \mathbf{1}_3 - \Pi_+, \end{aligned} \tag{16}$$

where $\mathbf{1}_3$ is an identity operator on Hilbert space of three qubits and

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{2\sqrt{3}}\left[(\sqrt{3}+1)|0\rangle_{\mathrm{s}}|01\rangle_{\mathrm{p}} - (\sqrt{3}-1)|0\rangle_{\mathrm{s}}|10\rangle_{\mathrm{p}} - 2\,|1\rangle_{\mathrm{s}}|00\rangle_{\mathrm{p}}\right], \\ |\phi_2\rangle &= \frac{1}{2\sqrt{3}}\left[(\sqrt{3}+1)|1\rangle_{\mathrm{s}}|10\rangle_{\mathrm{p}} - (\sqrt{3}-1)|1\rangle_{\mathrm{s}}|01\rangle_{\mathrm{p}} - 2\,|0\rangle_{\mathrm{s}}|11\rangle_{\mathrm{p}}\right]. \end{aligned} \tag{17}$$

Here the subscripts s and p label the states of signal and program qubits, respectively. After some algebra, we find the effective POVM carried out on the signal qubit,

$$\begin{aligned} \Pi'_{||} &= \frac{3-\sqrt{3}}{6}\mathbf{1} + \frac{\sqrt{3}}{3}|\psi\rangle\langle\psi|, \\ \Pi'_{\perp} &= \frac{3-\sqrt{3}}{6}\mathbf{1} + \frac{\sqrt{3}}{3}|\psi_\perp\rangle\langle\psi_\perp|. \end{aligned} \tag{18}$$

This POVM is symmetric (reflecting the symmetry of the program $|\psi\rangle|\psi_\perp\rangle$). The fidelity $F(\psi)$ is state independent and equal to the mean fidelity

$$F' = \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right). \tag{19}$$

Notice that $F' = \mathcal{F}_\perp$. This is not a mere coincidence, the optimum strategy for program $|\psi\rangle|\psi_\perp\rangle$ is to carry out an optimal estimation of $|\psi\rangle$ and then measure the signal qubit in the basis formed by estimated state $|\psi_{\mathrm{est}}\rangle$ and its orthogonal counterpart. The POVM (16) is an explicit implementation of this procedure. We emphasize here that F' is a maximum fidelity attainable with program $|\psi\rangle|\psi_\perp\rangle$, because the corresponding POVM solves the extremal equations. With the program $|\psi\rangle|\psi\rangle$ we achieve the fidelity $5/6 \approx 0.8333$ which is higher than F', hence the program $|\psi\rangle|\psi\rangle$ exhibits better performance than $|\psi\rangle|\psi_\perp\rangle$.

5 Possible experimental realization

The proposed universal measurement devices can be, at least in principle, realized experimentally. For example, the simplest one that uses a single-qubit program register only can be built up from one Fredkin (controlled swap) and two Hadamard gates. Signal and program enter the Fredkin gate as "controlled" qubits. The result can be read out from the ancillary qubit serving as a "control" qubit (the ancilla is originally in state $|0\rangle$). There are several ways how to implement this scheme on real physical systems [9]. For example, one can employ an experimental setup suggested previously in cavity-QED experiments [10].

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