

Detecting entanglement in two mode squeezed states by particle counting

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We present an entanglement criterion for two mode squeezed states which relies on particle counting only. The proposed inequality is optimal for the state under consideration and robust against particle losses up to 2/3. As it does not involve measurements of quadratures - which is typically very challenging for atomic modes - it renders the detection of atomic many-particle entanglement feasible in many different settings. Moreover it bridges the gap between entanglement verification for a qubit and criteria for continuous variables measured by homodyne detection. We illustrate its application in the context of superradiant light scattering from Bose Einstein condensates by considering the creation of entanglement between atoms and light as well as between two condensates in different momentum states. The latter scheme takes advantage of leaving the Gaussian realm and features probabilistic entanglement distillation.

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I. INTRODUCTION

Entanglement is a true quantum feature. The study of this peculiarity of physics does not only hold the promise to acquire a deeper understanding of Nature, but also paves the way towards auspicious applications of quantum information science such as unconditionally secure communication, ultraprecise measurements, quantum computing and quantum simulation. Therefore, the quest for entanglement or inseparability criteria is a vigorous field of research [1].

In the most basic case, entanglement is shared between two parties holding a single particle each and bipartite entanglement of single pairs is well understood. Then again, entanglement between parties holding a huge number of particles was studied for Gaussian states with great success [2]. Here, the natural question arises, how entanglement can be verified in the intermediate regime and how the two well-studied cases of single pairs and Gaussian modes can be linked by an inseparability criterion for an arbitrary number of particles, which is not restricted to Gaussian states. Starting from this motivation, we bridge the gap between inseparability criteria for a qubit and continuous variables for entanglement in two mode squeezed states. Moreover, from a practical point of view, the new entanglement condition provides experimental feasibility and applicability in numerous settings.

More specifically, entanglement in two mode squeezed states can, in principle, be detected by means of a Gaussian inseparability criterion [3], which requires the measurement of variances in canonical quadratures. This can be conveniently performed for light modes, as first demonstrated in [4], as well as for multi-atom collective spin modes, as shown in [5]. However, in many cases involving multi-atom entanglement homodyne measurement of atomic canonical variables which require an atomic "local oscillator" is not feasible.

This problem can be overcome by means of the practical entanglement condition put forward in this work.

The proposed inequality requires only particle number measurements, rather than measurement of quadratures and can be used to detect N -particle entangled states of the form $|TMSS\rangle \otimes |TMSS\rangle$, which is attractive in view of many recent experiments, that offer the potential to generate this type of entanglement.

For example, superradiant scattering [6] of laser light from a Bose-Einstein condensate was observed recently [7]. Superradiant scattering leads to highly directional emission of light from the atomic sample. This striking effect attracted considerable interest and it was shown theoretically that two mode squeezed states can be generated in this context [8, 9, 10]. This is particularly interesting, since it represents the interspecies atom-light analogue of photonic twin-beams generated in optical parametric down conversion, which plays an essential role in many applications of quantum optics and quantum information theory. Despite the fact that the dynamics of the process and the resulting non-separable state are well understood and the system is known to be a very promising candidate for entanglement generation [8, 9, 10, 11], entanglement could not be verified in this system due to the absence of a suitable inseparability criterion. Thus, nonclassical correlations have been studied, but since the quantum states produced in an experiment cannot be assumed to be pure, correlations do not imply the presence of entanglement. Other examples can be found in many different setups, as for example in entanglement production in spin exchange collisions in Bose-Einstein condensates [12], the free electron laser [13], the creation of two mode squeezed states by dissociation [14] and entanglement production in colliding Bose-Einstein condensates [15] or four-wave mixing in matter waves in an optical lattice [16]. While correlations could be observed, multi-particle entanglement has not yet been detected in this context. This gap can be closed by means of the entanglement criterion put forward in this work. It is robust against the sources of noise to be expected in a realistic setup. As the Gaussian criterion [3], it can be seen as a

local uncertainty relation [17], with the difference that it involves a bound, which is given by expectation values of operators, rather than uncertainty limits represented by fixed numbers. The presented inequality is optimal for the state under consideration and provides a possibility for successful detection of entanglement even for highly mixed states.

The inseparability criterion is presented and proven in section II. In this section we also consider the influence of particle losses and find that for symmetric particle losses in Alice's and Bob's systems, loss of a fraction up to 2/3 of all particles can be tolerated. In section III we illustrate the application of the inseparability criterion by means of two specific examples in the context of super-radiant light scattering from Bose-Einstein condensates. More precisely, we consider the creation of entanglement between atoms and light in superradiant Raman scattering and describe a scheme in which entanglement between a moving condensate and a condensate at rest is created and purified.

II. INSEPARABILITY CRITERION BASED ON PARTICLE NUMBER MEASUREMENTS

As outlined above, the proposed entanglement criterion is optimal for the state $|TMS\rangle \otimes |TMS\rangle$. In the Fock basis this quantum state is given by

$$|\Psi_1\rangle = (1 - \Lambda^2) \sum_{n=0}^{\infty} \Lambda^n |n\rangle |n\rangle \otimes \sum_{m=0}^{\infty} \Lambda^m |m\rangle |m\rangle, \quad (1)$$

where $\Lambda = \tanh(r)$ and $r \in \mathbb{C}$ is the squeezing parameter. The second and fourth ket refer to Alice's system, which is described by two modes with creation operators a_+^\dagger and a_-^\dagger . Likewise, the first and third ket refer to Bob's system, which is described by two modes with creation operators b_+^\dagger and b_-^\dagger . Using this notation, we change to the Schwinger representation and define Stokes operators S_x , S_y and S_z for Alice's system and J_x , J_y and J_z for Bob's. \mathbf{S} is given by

$$\begin{aligned} S_x &= (n_{A,x} - n_{A,y})/2, \\ S_y &= (n_{A,+45} - n_{A,-45})/2, \\ S_z &= (n_{A,+} - n_{A,-})/2, \end{aligned} \quad (2)$$

and \mathbf{J} is defined by analogous expressions. The number operators n carry subscripts for Alice's/Bob's system (A/B) and for the three different bases (x/y , $+45/-45$, $+/-$), where $n_{A,\pm} = a_\pm^\dagger a_\pm$ and $n_{B,\pm} = b_\pm^\dagger b_\pm$.

A. Entanglement criterion

The main characteristic feature of the two mode squeezed state $|\Psi_1\rangle$ is the correlation of particle numbers in Alice's and Bob's system. In contrast, any separable state satisfies a lower bound for the difference in

particle numbers in different mutually independent bases. More precisely, any 2×2 - mode bipartite separable state, $\rho = \sum_i p_i \rho_i$, $\rho_i = \rho_i^A \otimes \rho_i^B$ (where $p_i \geq 0$ and $\sum_i p_i = 1$), satisfies the inequality

$$\begin{aligned} &\langle (J_x - S_x)^2 \rangle_\rho + \langle (J_y + S_y)^2 \rangle_\rho + \langle (J_z - S_z)^2 \rangle_\rho \\ &\geq (\langle n_A \rangle_\rho + \langle n_B \rangle_\rho) / 2. \end{aligned} \quad (3)$$

In the following, we prove this entanglement criterion. For any ρ_i , the left side of inequality (3) equals

$$\begin{aligned} &\langle J_x^2 + J_y^2 + J_z^2 \rangle_{\rho_i} + \langle S_x^2 + S_y^2 + S_z^2 \rangle_{\rho_i} \\ &- 2 \langle J_x S_x - J_y S_y + J_z S_z \rangle_{\rho_i}. \end{aligned} \quad (4)$$

$\langle J_x^2 + J_y^2 + J_z^2 \rangle_{\rho_i} = \langle (n_B/2) ((n_B/2) + 1) \rangle_{\rho_i}$, where $n_B = n_{B,+} + n_{B,-}$, and an analogous equality holds for the second term, as can be inferred from definition (2). Since ρ_i is assumed to be separable, the third term in expression (4) can be expressed as a product of two expectation values

$$-2 \langle \mathbf{J} \tilde{\mathbf{S}} \rangle_{\rho_i} = -2 \langle \mathbf{J} \rangle_{\rho_i} \langle \tilde{\mathbf{S}} \rangle_{\rho_i}$$

where $\tilde{\mathbf{S}} = (S_x \ S_y \ S_z)^T$. Using $\langle \mathbf{J} \rangle_{\rho_i} \leq \langle n_B/2 \rangle_{\rho_i}$ and $\langle \tilde{\mathbf{S}} \rangle_{\rho_i} \leq \langle n_A/2 \rangle_{\rho_i}$, we obtain that expression (4) is greater than or equal to

$$\left\langle \frac{n_B}{2} \left(\frac{n_B}{2} + 1 \right) \right\rangle_{\rho_i} + \left\langle \frac{n_A}{2} \left(\frac{n_A}{2} + 1 \right) \right\rangle_{\rho_i} - 2 \left\langle \frac{n_B}{2} \right\rangle_{\rho_i} \left\langle \frac{n_A}{2} \right\rangle_{\rho_i}.$$

Since $\langle n_A/2 \rangle_{\rho_i} \langle n_B/2 \rangle_{\rho_i} = \langle n_A n_B / 4 \rangle_{\rho_i}$ for product states, we can reexpress this equation by

$$\left\langle \left(\frac{n_B}{2} - \frac{n_A}{2} \right)^2 \right\rangle_{\rho_i} + \left\langle \frac{n_B}{2} + \frac{n_A}{2} \right\rangle_{\rho_i} \geq \left\langle \frac{n_B}{2} + \frac{n_A}{2} \right\rangle_{\rho_i}.$$

As this is true for every ρ_i , the average $\langle (n_A + n_B) / 2 \rangle_\rho$ is a lower bound for the mixture $\rho = \sum_i p_i \rho_i$.

This limitation imposed on convex mixtures of product states can be overcome if entanglement is involved. In particular,

$$\begin{aligned} &\langle (J_x - S_x)^2 \rangle_{|\Psi_1\rangle \langle \Psi_1|} + \langle (J_y + S_y)^2 \rangle_{|\Psi_1\rangle \langle \Psi_1|} \\ &+ \langle (J_z - S_z)^2 \rangle_{|\Psi_1\rangle \langle \Psi_1|} = 0, \end{aligned}$$

as the two mode squeezed state $|\Psi_1\rangle$ is a simultaneous eigenstate of $(J_x - S_x)$, $(J_y + S_y)$ and $(J_z - S_z)$ with common eigenvalue 0.

B. Implications of particle losses

In this subsection, we analyze how particle losses impair the performance of the presented entanglement criterion. The influence of particle losses is modeled by a beam splitter transformation. Creation operators for atoms and light transform according to

$$\begin{aligned} a_\pm^\dagger &\mapsto \sqrt{1 - r_A} a_\pm^\dagger - i \sqrt{r_A} v_{A\pm}^\dagger, \\ b_\pm^\dagger &\mapsto \sqrt{1 - r_B} b_\pm^\dagger - i \sqrt{r_B} v_{B\pm}^\dagger, \end{aligned}$$

corresponding to a beamsplitter with reflectivity r_A for Alice's system and r_B for Bob's system. The quantum noise operators $v_{A\pm}^\dagger$, and $v_{B\pm}^\dagger$ obey canonical commutation relations for each mode and are mutually independent. For the target state $|\Psi_1\rangle$, the left side of condition (3) is transformed into

$$\frac{3}{2} \left((r_A - r_B)^2 (\Delta n)^2 + (r_A(1 - r_A) + r_B(1 - r_B)) \langle n \rangle \right), \quad (5)$$

where $\langle n \rangle = \sinh^2(r)$ and $(\Delta n)^2 = 2 \sinh^4(r)$ is the variance of the particle number [20]. By applying the same beamsplitter transformation to the right side of inequality (3) one obtains

$$(1 - r_A) \langle n \rangle + (1 - r_B) \langle n \rangle.$$

In case of symmetric losses $r_A = r_B = r$, successful entanglement verification requires therefore $r < 2/3$. Limitations imposed by particle losses which are different for Alice's and Bob's system are more restrictive, as they impair directly the symmetry-property to which the criterion is tailored to. For large particle numbers, the first term in expression (5) is likely to hinder the detection of entanglement. This problem can be resolved by introducing gain factors g_A and g_B for Alice and Bob, which characterize the amplification of measured signals

$$\begin{aligned} a_\pm^\dagger &\mapsto \sqrt{g_A(1 - r_A)} a_\pm^\dagger - i\sqrt{r_A} v_{A\pm}^\dagger, \\ b_\pm^\dagger &\mapsto \sqrt{g_B(1 - r_B)} b_\pm^\dagger - i\sqrt{r_B} v_{B\pm}^\dagger. \end{aligned}$$

In this situation, we obtain the result

$$\begin{aligned} &\frac{3}{2} (g_B(1 - r_B) - g_A(1 - r_A))^2 (\Delta n)^2 + \\ &\frac{3}{2} (g_B r_B(1 - r_B) + g_A r_A(1 - r_A)) \langle n \rangle, \end{aligned}$$

which has to be compared to

$$g_B(1 - r_B) \langle n \rangle + g_A(1 - r_A) \langle n \rangle.$$

g_B and g_A have to be optimized according to the experimental parameters. For large particle numbers, the quadratic term will dominate such that $g_A/g_B = (1 - r_B)/(1 - r_A)$ renders the problem of asymmetric losses and we obtain the condition $(r_A + r_B)/2 < 2/3$ for successful entanglement detection. Remarkably this threshold does not depend on the degree of squeezing. If the probabilities for particle losses are known, atomic and photonic signals need not to be amplified, as it is sufficient to adjust the inequality accordingly.

III. ENTANGLEMENT IN SUPERRADIANT SCATTERING FROM BOSE-EINSTEIN CONDENSATES

As explained above, the presented entanglement criterion can be applied in many different settings. We describe here the verification of entanglement produced in

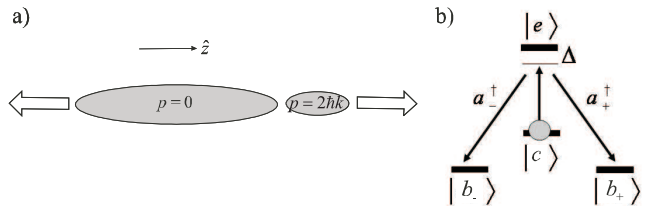


FIG. 1: Setup and atomic levels considered for the creation of N -particle entanglement between atoms and light. a) Light propagating along \hat{z} is scattered from an elongated condensate. Photons are scattered superradiantly into two endfire-modes, which correspond to scattering angles of 0 and π , as indicated by arrows. b) Off-resonant laser light couples to the transition $|c\rangle \rightarrow |e\rangle$, such that atoms, which are initially prepared in $|c\rangle$ are transferred to state $|b_+\rangle$ or $|b_-\rangle$ via a Raman process and emit a photon in $+$ or $-$ polarization respectively.

superradiant scattering of laser light from a Bose Einstein condensate. More specifically, we consider a Bose Einstein condensate, which is elongated along \hat{z} and excited by a laser field propagating along the same direction. The scattering interaction is assumed to be well within the superradiant regime, such that light is predominantly emitted along two endfire-modes [8], along \hat{z} and $-\hat{z}$, as shown in figure 1a. Atoms scattering photons along $-\hat{z}$ acquire a momentum kick of $2\hbar k$, where k is the wave vector of the incoming light field. These atoms get spatially separated from the BEC and form a new moving condensate. To begin with, only the endfiremode consisting of photons scattered along $-\hat{z}$ and atoms traveling along the \hat{z} direction are considered.

A. Entanglement between atoms and light

Atoms are assumed to possess an internal level structure as shown in figure 1b. As was shown in [9], the dynamics of the superradiant process can be described by a two mode squeezing Hamiltonian

$$H_1 \propto a_+^\dagger b_+^\dagger + a_-^\dagger b_-^\dagger + H.C.,$$

where the creation operators a_+^\dagger and a_-^\dagger denote the scattered light fields in plus and minus polarization, while b_+^\dagger and b_-^\dagger are the creation operators for the respective atomic states. This leads to the generation of the two mode squeezed state $|\Psi_1\rangle$. Atom and photon numbers are correlated for each polarization and inseparability of the produced quantum state can be verified according to criterion (3) by identifying Alice with the light field and Bob with the atomic system. In the considered physical setting, various sources of noise may impair the verification of entanglement. Apart from particle losses, which have been discussed in the previous section, undesired atomic transitions can degrade the reliability of the proposed criterion, for instance when atoms are scattered

into states other than $|b_-\rangle$ and $|b_+\rangle$, while emitting photons in $+$ or $-$ polarization. These processes can be avoided by a suitable choice of atomic levels. As an example we consider typical Alkali atoms used in BEC experiments, ^{87}Rb and ^{23}Na , which have nuclear spin $3/2$. By preparing the atomic sample in the $F = 1$, $m_F = 0$ ground state and inducing transitions to a manifold with $F' = 0$, $m_{F'} = 0$, atoms can only be scattered back to the $F = 1$ groundstate manifold, occupying the states $|F = 1, m_F = -1\rangle \equiv |b_-\rangle$ and $|F = 1, m_F = +1\rangle \equiv |b_+\rangle$, while transitions to other states are forbidden due to the selection rule $\Delta(F) = 1$. Unintentional transitions may also be mediated by interatomic collisions. The effect of transitions from $|b_+\rangle$ or $|b_-\rangle$ to other states is already included in consideration of particle losses above, but the creation of a pair of atoms in these two states without the production of the corresponding photon pair has to be avoided. This can be done by applying electromagnetic fields imposing Stark shifts on the internal levels such that such a transition is prohibited by energy conservation.

The measurement of Stokes operators of light required for the verification of the entanglement can be performed in a standard fashion [18]. The measurement of the atomic collective spin projection J_z can be done by counting atoms in the final states $+, -$ with resonant absorptive imaging. The measurement of the projections $J_{x,y}$ can be performed by applying suitable radio-frequency $\pi/2$ pulses to the final atomic states and then doing absorptive imaging and atom counting.

B. Entanglement between two condensates

The correlations between atoms and light, that are generated in the process described above, can be used to create entanglement between two condensates. To this end, both endfire modes are considered. The full Hamiltonian is given by

$$H_2 \propto a_{+I}^\dagger b_{+I}^\dagger + a_{-I}^\dagger b_{-I}^\dagger + a_{+II}^\dagger b_{+II}^\dagger + a_{-II}^\dagger b_{-II}^\dagger + H.C. ,$$

where the subscript I refers to the backward scattered light and the moving condensate, while the subscript II refers to forward scattered light and the condensate at rest. After the interaction, the backward scattered light field and the moving condensate are in a two mode squeezed state as well as the light field scattered in forward direction and the part of the condensate at rest, which is transferred to state $|b_+\rangle$ or $|b_-\rangle$,

$$|\Psi_2\rangle = (1 - \Lambda)^4 \sum_{n=0}^{\infty} \Lambda^n |n\rangle |n\rangle \otimes \sum_{j=0}^{\infty} \Lambda^j |j\rangle |j\rangle \\ \otimes \sum_{m=0}^{\infty} \Lambda^m |m\rangle |m\rangle \otimes \sum_{l=0}^{\infty} \Lambda^l |l\rangle |l\rangle,$$

where the first and second term refer to I -operators - atom-photons pairs in plus and minus polarization re-

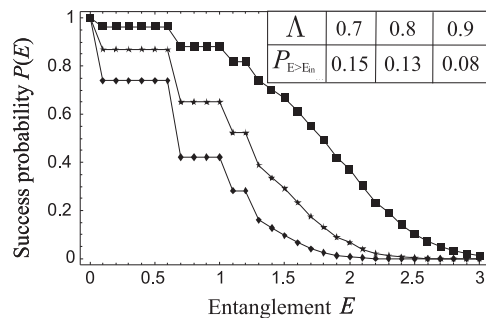


FIG. 2: Success probability versus entanglement which can be produced by means of the proposed scheme for different values of Λ . Diamonds: $\Lambda = 0.7$, stars: $\Lambda = 0.8$, squares: $\Lambda = 0.9$. The inset shows the probability of obtaining at least as much entanglement as was present in the input state $|\Psi_1\rangle$.

spectively - while the third and fourth term refer to II -operators.

The part of the resting condensate being in state $|b_+\rangle$ or $|b_-\rangle$ can be entangled with the moving condensate by means of entanglement swapping, i.e. by measuring EPR operators for each polarization of light modes using homodyne detection. However, this procedure leads to degradation of entanglement if non-maximally entangled states are involved, and a distillation step has to be performed afterwards to obtain a more useful resource state. It has been shown that continuous variable entanglement cannot be distilled using only Gaussian operations [19] [21]. In the following, we describe therefore a scheme which relies on photon counting rather than Gaussian measurements and exhibits probabilistic entanglement distillation. More specifically, the moving and resting condensates are entangled by combining the forward and backward scattered light modes at a beamsplitter and measuring the photon numbers at both output ports. This has to be done for each polarization separately. We explain the scheme for the $+$ polarized part of the light field. Analogous expressions hold for the $-$ polarized part.

By applying a balanced beamsplitter transformation $a_{+I}^{in} \rightarrow (a_{+I}^{out} + a_{+II}^{out})/\sqrt{2}$, $a_{+II}^{in} \rightarrow (a_{+I}^{out} - a_{+II}^{out})/\sqrt{2}$, where a_{+I}^{in}/a_{+II}^{in} and $a_{+I}^{out}/a_{+II}^{out}$ denote annihilation operators of the light fields at the input and output ports of the beamsplitter respectively, to state (1), we obtain

$$|\Psi_1^{BS}\rangle = (1 - \Lambda^2) \sum_{n,m=0}^{\infty} \Lambda^{n+m} \frac{1}{\sqrt{n!m!}} 2^{-\frac{(n+m)}{2}} \sum_{i=0}^n \sum_{j=0}^m \\ \binom{n}{i} \binom{m}{j} (-1)^j \sqrt{(i+j)!(n+m-i-j)!} \\ |n, i+j\rangle |m, n+m-i-j\rangle.$$

The probability of detecting N_I photons at the first, and N_{II} photons at the second output port of the beamsplitter is $P_{N_I, N_{II}} = (1 - \Lambda^2) \Lambda^{N_I + N_{II}}$. Such an event results

in the quantum state

$$|\Psi_{N_I, N_{II}}\rangle = \sum_{n=0}^N k_{N_I, N_{II}}(n) |n, N-n\rangle,$$

where $N = N_I + N_{II}$ denotes the total number of detected photons. The coefficients $k_{N_I, N_{II}}(n)$ are given by

$$k_{N_I, N_{II}}(n) = 2^{\frac{-N}{2}} (-1)^{N_I} \sqrt{\frac{N_{II}!}{N_I!}} \sqrt{\frac{(N-n)!}{n!}} \frac{1}{(N_{II}-n)!} {}_2F_1(-n, -N_I, N_{II}-n+1; -1)$$

where ${}_2F_1(a, b, c; z)/(c-1)!$ is the regularized hypergeometric function. This state describes now pairs of atoms in $|b_+\rangle$ in the moving condensate and at rest, which are referred to in the first and second ket respectively. For certain measurement outcomes this state is more entangled than $|\Psi_1\rangle$, such that the state can be purified by postselection. Figure 2 shows the success probability versus the produced entanglement given by the von Neumann entropy of the reduced density matrix of the resulting atomic state $E(N_I, N_{II}) = \sum_{n=0}^{\infty} k_{N_I, N_{II}}^2(n) \ln(k_{N_I, N_{II}}^2(n))$ for different values of Λ . Note that the initial state, which contains infinitely many terms, is truncated by the measurement process. In this way states with entanglement close to the maximal degree of entanglement in the corresponding subspace can be produced. For example, for $N_I = 1$ and $N_{II} = 0$, the maximally entangled state $|\Psi\rangle = (|1, 0\rangle + |1, 0\rangle)/\sqrt{2}$ is created. Considering the light field in $+$ as well as in $-$ polarization, the resulting atomic state can be detected by criterion (3) after local transformation $n_{+II} \mapsto$

$N - n_{+II}$ and $n_{-II} \mapsto N' - n_{-II}$. In this case \mathbf{J} refers to atomic operators at rest and \mathbf{S} describes the moving condensate.

IV. CONCLUSIONS

In conclusion, we presented a practical criterion for the verification of entanglement in two mode squeezed states, which is solely based on particle number measurements. As it does not involve measurement of canonical variables, it is applicable in a wide variety of settings. We expect the presented inequality to be especially useful in experiments involving atomic modes and illustrate a specific example by considering superradiant light scattering from Bose Einstein condensates. In contrast to Gaussian inseparability criteria, the entanglement condition put forward in this work is valid for an arbitrary number of particles. It connects therefore the verification of entanglement in two mode squeezed states in the realm of continuous variables and the single particle case.

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